1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

2. Prove that there is no rational number whose square is 12.
Proof:

3. Prove Proposition 1.15.
(a) If $x \neq 0$, and $xy = xz$, then $y = z$.
(b) If $x \neq 0$, and $xy = x$, then $y = 1$.
(c) If $x \neq 0$, and $xy = 1$, then $y = 1/x$.
(d) If $x \neq 0$, then $1/(1/x) = x$.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf(A) = -\sup(-A)$.

6. Fix $b > 1$.
(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that $\left(b^m\right)^{\frac{1}{n}} = \left(b^p\right)^{\frac{1}{q}}$.
Hence it makes sense to define $b^r = \left(b^m\right)^{\frac{1}{n}}$.
(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c) If x is real, define $B(x)$ to be the set of all number b^t, where t is rational and $t \leq x$.
Prove that $b^r = \sup B(r)$ whene r is rational.
Hence it makes sense to define $b^x = \sup B(x)$ for every real x.
(d) Prove that $b^{x+y} = b^x b^y$.

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$.
(This is called the logarithm of y to the base b.)
Complete the following outline.
(a) For any positive integer n, $b^n - 1 \geq n(b - 1)$.
(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.
(c) If $y > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.
(d) If w is such that $b^w < y$, then $b^{w+(1/n)} > y$ for sufficiently large n.
(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.
(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup(A)$ satisfies $b^x = y$.
(g) Prove that this x is unique.