Let (X, d) be a metric space.

1. Let $\{x_n\} \subset X$ and $0 < r < 1$. Suppose that $d(x_{n+1}, x_n) \le r \bullet d(x_n, x_{n-1})$, $\forall\, n \ge 2$. Show that $\{x_n\}$ is a Cauchy sequence.

2. Let $p \in \mathbb{N}$ and $\{x_n\}$ be defined recursively by $x_1 = p$ and $x_{n+1} = \dfrac{x_n}{2} + \dfrac{p}{x_n}$, $\forall\, n \ge 1$.
Show that $\{x_n\}$ converges, and find the limit of the sequence.

3. Let $x_n = \sum_{k=1}^{n} \dfrac{2^k}{k^2 3^k}$, $\forall\, n \in \mathbb{N}$. Show that $\{x_n\}$ is convergent.

4. Let $x_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n}$, $\forall\, n \in \mathbb{N}$. Show that $\{x_n\}$ is convergent.

5. Let $x_1 = 1$, $x_n = 3 - \dfrac{1}{x_{n-1}}$, $\forall\, n \ge 2$. Show that $\{x_n\}$ is convergent and find $\lim\limits_{n\to\infty} x_n$.

6. Let $x_1 = a$, $x_2 = b$ and $a < b$. Define $x_{n+2} = (1/2)(x_{n+1} + x_n)$, $\forall\, n \ge 1$.
Show that $\{x_n\}$ is convergent and find $\lim\limits_{n\to\infty} x_n$.

7. Determine whether or not the following sequences converge. Find the limits of the sequences that converge.

(a) $x_n = \dfrac{2n^2 - 1}{3n^2 + n + 5}$;

(b) $x_n = \sqrt{n^2 + n} - \sqrt{n^2 - n}$;

(c) $x_n = \dfrac{n!}{n^n}$; =

(d) $x_n = \dfrac{1 \bullet 3 \bullet 5 \bullet \cdots \bullet (2n-1)}{n!}$;

(e) $x_n = \dfrac{3^n + n^2}{4^n + n}$;

(f) $x_n = \sum_{k=1}^{n} \dfrac{1}{\sqrt{n^2 + k}}$;

8. Let $\{x_n\}$ and $\{y_n\}$ be two bounded real sequences. Show that:

(a)

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) Show that the inequalities in (a) might be strict.

(c) Show that if $x_n \to x$, then

$$\liminf_{n\to\infty}(x_n + y_n) = x + \liminf_{n\to\infty} y_n \text{ and } \limsup_{n\to\infty}(x_n + y_n) = x + \limsup_{n\to\infty} y_n.$$

(d) If $x_n \ge 0$ and $y_n \ge 0$, $\forall\, n \in \mathbb{N}$, then

$$\liminf_{n\to\infty} x_n \bullet \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty}(x_n \bullet y_n) \le \limsup_{n\to\infty}(x_n \bullet y_n) \le \limsup_{n\to\infty} x_n \bullet \limsup_{n\to\infty} y_n.$$

Show that these inequalities might be strict.

(e) If $x_n \ge 0$, $y_n \ge 0$ $\forall\, n \in \mathbb{N}$ and $x_n \to x > 0$, then

$$\liminf_{n\to\infty}(x_n \bullet y_n) = x \bullet \liminf_{n\to\infty} y_n \text{ and } \limsup_{n\to\infty}(x_n \bullet y_n) = x \bullet \limsup_{n\to\infty} y_n.$$

9. Let $\{x_n\}$ be a bounded sequence of real numbers. Show that:

(a) $\liminf_{n\to\infty} x_n = \alpha \Leftrightarrow \forall\, \varepsilon > 0$ there are infinitely many terms of $\{x_n\}$ in $(\alpha - \varepsilon, \alpha + \varepsilon)$, but only finitely many in $(-\infty, \alpha - \varepsilon)$.

(b) $\limsup_{n\to\infty} x_n = \beta \Leftrightarrow \forall\, \varepsilon > 0$ there are infinitely many terms of $\{x_n\}$ in $(\beta - \varepsilon, \beta + \varepsilon)$, but only finitely many in $(\beta + \varepsilon, +\infty)$.

10. Let $\{x_n\} \subset \mathbb{R}$ and $s_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$

(a) Show that $\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} x_n$.

(b) Show that if $x_n \to x$, then $s_n \to x$.

(c) Give an example of divergent $\{x_n\}$ such that $\{s_n\}$ is convergent.