11. Solve the following exercises on pages 78 - 82:
1-5, 14, 16, 17, 20, 21, 23.

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

2. Calculate $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

3. If $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ $(n = 1, 2, \ldots)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, \ldots$.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by $s_1 = 0$; $s_{2m} = \frac{s_{2m-1}}{2}$; and $s_{2m+1} = \frac{1}{2} + s_{2m}$.

5. For any two real sequence $\{a_n\}$ and $\{b_n\}$, prove that $\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided the sum on the right is not of the form $\infty - \infty$.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic mean σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots)$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.
(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?
(d) Put $a_n = s_n - s_{n-1}$, for $n \ge 1$. Show that $s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$.
Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]
(e) Derive the last conclusion (Prove that $\{s_n\}$ converges.) from a weaker hypothesis: Assume $M < \infty$, $|na_n| \le M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:
(i) If $m < n$, then $s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i)$.
(ii) For these i, $|s_n - s_i| \le \frac{(n-i)M}{i+1} \le \frac{(n-m-1)M}{m+2}$.
(iii) Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies $m \le \frac{n-\varepsilon}{1+\varepsilon} < m+1$.
(iv) Then $(m+1)/(n-m) \le 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$.
(v) Hence $\limsup_{n\to\infty} |s_n - \sigma| \le M\varepsilon$. Since ε was arbitrary, $\lim s_n = \sigma$.

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$ and define $x_2, x_3, \ldots$, by the recursion formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$.

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$ so that, setting $\beta = 2\sqrt{\alpha}$, we have $\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$ $(n = 1, 2, 3, \ldots)$.

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < 1/10$ and that therefore $\varepsilon_5 < 4 \bullet 10^{-16}$, $\varepsilon_6 < 4 \bullet 10^{-32}$.

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define $x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}$.

(a) Prove that $x_1 > x_3 > x_5 > \ldots$

(b) Prove that $x_2 < x_4 < x_6 < \ldots$

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in Exercise 16.

This algorithm converges much slower than the one described in Exercise 16. For $\alpha = 3$ and $x_1 = 2$, the error of each x_n is at least $1/10^{n-1}$.

Substituting 3 for α and 2 for x_1, we have $\varepsilon_{2k+1} > \left(2-\sqrt{3}\right)\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)^{2k} > 1 \bullet \frac{1}{10^{2k-1}}$.

And $\left|\varepsilon_{2k}\right| > \varepsilon_{2k}\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right) > \left(2-\sqrt{3}\right)\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)^{2k} > 1 \bullet \frac{1}{10^{2k-1}}$, $\forall\, k \in \mathbb{N}$.

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint*: For any m, n,
$d(p_n, q_n) \le d(p_n, p_m) + d(p_n, q_m) + d(q_m, q_n)$; it follows that $|d(p_n, q_n) - d(p_m, q_m)|$ is small if m and n are large.