Rudin, Chapter 3, pp. 78 - 79, #6, 7, 8, 11, 12

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if
(a) $a_n = \sqrt{n+1} - \sqrt{n}$;
(b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;
(c) $a_n = \left(\sqrt[n]{n} - 1\right)^n$;
(d) $a_n = \dfrac{1}{1+z^n}$ for complex values of z.

7. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum_{n=0}^{\infty} \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.
(a) Prove that if $\sum a_n$ diverges $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges.
(b) Assume $a_n > 0$ and $\sum a_n$ diverges. Let $s_n = \sum_{i=1}^{n} a_i$. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges.

Hint: $\dfrac{a_{N+1}}{s_{N+1}} + \cdots + \dfrac{a_{N+k}}{s_{N+k}} \geq 1 - \dfrac{s_N}{s_{N+k}}$.
(c) Assume $a_n > 0$ and $\sum a_n$ diverges. Let $s_n = \sum_{i=1}^{n} a_i$. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ converges.

Hint: $\dfrac{a_n}{s_n^2} \leq \dfrac{1}{s_{n-1}} - \dfrac{1}{s_n}$.
(d) What can be said about $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$?

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum_{m=n}^{\infty} a_m$.
(a) Prove that $\dfrac{a_m}{r_m} + \cdots + \dfrac{a_n}{r_n} > 1 - \dfrac{r_n}{r_m}$ if $m < n$, and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.

(b) Prove that $\dfrac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$ and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ converges.