Math 230A, Practice Problems, Assigned 9/17/10

1. Write detailed proofs of Theorems 1.31 and 1.33.

Theorem 1.31 If z and w are complex, then

(a) $\overline{z+w} = \bar{z} + \bar{w}$,

(b) $\overline{zw} = \bar{z} \bullet \bar{w}$,

(c) $z + \bar{z} = 2\text{Re}(z)$, $z - \bar{z} = 2i\,\text{Im}(z)$,

(d) $z\bar{z}$ is real and positive (except when z = 0).

Theorem 1.33 Let z and w be complex numbers. Then

(a) $|z| > 0$ unless $z = 0$, $|0| = 0$,

(b) $|\bar{z}| = |z|$,

(c) $|zw| = |z||w|$,

(d) $|\text{Re}\, z| \le |z|$,

(e) $|z + w| \le |z| + |w|$.

2. Read Definition 1.36 of the vector space $\mathbb{R}^n$. Write detailed proof of Theorem 1.37.

Theorem 1.37 Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then

(a) $|\mathbf{x}| \ge 0$;

(b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = 0$;

(c) $|\alpha\mathbf{x}| = |\alpha||\mathbf{x}|$;

(d) $|\mathbf{x} \bullet \mathbf{y}| \le |\mathbf{x}| \bullet |\mathbf{y}|$;

(e) $|\mathbf{x} + \mathbf{y}| \le |\mathbf{x}| + |\mathbf{y}|$;

(f) $|\mathbf{x} - \mathbf{y}| \le |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

3. Repeat Ex. 2 for $\mathbb{C}^n$. Note $z \bullet v = \sum_{i=1}^{n} z_i \bullet \bar{v}_i$ and $|z| = \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2}$.

Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n$, and α is real. Then

(a) $|\mathbf{x}| \ge 0$;

(b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = 0$;

(c) $|\alpha\mathbf{x}| = |\alpha||\mathbf{x}|$;

(d) $|\mathbf{x} \bullet \mathbf{y}| \le |\mathbf{x}| \bullet |\mathbf{y}|$;

(e) $|\mathbf{x} + \mathbf{y}| \le |\mathbf{x}| + |\mathbf{y}|$;

(f) $|\mathbf{x} - \mathbf{y}| \le |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

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4. In the previous exercises the norm of a vector is defined by an inner product. We can define norms in $\mathbb{R}^n$ (or $\mathbb{C}^n$) without the use of an inner product, as a function "$\bullet$": $\mathbb{R}^n \to [0, +\infty)$ with the properties (a), (b), (c), and (e) from Theorem 1.37.
Example: Consider an arbitrary, but fixed $p \geq 1$.
Define $|\mathbf{x}|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, for $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{C}^n$. Show that $|\mathbf{x}|_p$ has the properties (a), (b), (c), (e).

Hint: For (e) first prove that $\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \bullet \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$ where $1/p + 1/q = 1$. This is called the Holder inequality.
Remark: Property (e) is called the Minkowski inequality and looks like

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

(a) $|\mathbf{x}|_p \geq 0$;
(b) $|\mathbf{x}|_p = 0$ if and only if $\mathbf{x} = 0$;
(c) $|\alpha \mathbf{x}|_p = |\alpha||\mathbf{x}|_p$;
(e) $|\mathbf{x} + \mathbf{y}|_p \leq |\mathbf{x}|_p + |\mathbf{y}|_p$;

5. One way to prove Holder's inequality is to use Young's inequality:
$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, $\forall\, a, b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove Young's inequality.

6. Show that $|\mathbf{x}|_p$ satisfies the parallelogram identity from Exercise 17/Chapter 1 if and only if $p = 2$.

7. Define $|\mathbf{x}|_\infty = \max_{1 \leq i \leq n} |x_i|$, for $\mathbf{x} \in \mathbb{R}^n$ (or $\mathbb{C}^n$). Prove that $|\mathbf{x}|_\infty$ satisfies the properties of the norm.
(a) $|\mathbf{x}|_\infty \geq 0$
(b) $|\mathbf{x}|_\infty = 0$ if and only if $\mathbf{x} = 0$
(c) $|\alpha \mathbf{x}|_\infty = |\alpha||\mathbf{x}|_\infty$
(e) $|\mathbf{x} + \mathbf{y}|_\infty \leq |\mathbf{x}|_\infty + |\mathbf{y}|_\infty$