Pages 4, 5, Assigned Friday, 9/24/10

Use Definition 2.3 to define $A \sim B \Leftrightarrow \exists f : A \to B$ that is 1-1 and onto.
A and B have the same cardinal number if $A \sim B$.
Use the notation card(A) for the cardinal number of A.
card(A) can be identified with the equivalence class of the set A: $[A] = \{\text{sets } B : A \sim B\}$.

Examples: card$\{1, 2, ..., n\} = n$, card($\mathbb{N}$) = card($\mathbb{Z}$) = card($\mathbb{Q}$) = $\aleph_0$, card($\mathbb{R}$) = $\mathfrak{c}$.

1. Define an order in the set of cardinal numbers:
card(A) ≤ card(B) $\Leftrightarrow \exists f : A \to B$ that is 1-1.

(a) Show that card(A) ≤ card(B) $\Leftrightarrow \exists g: B \to A$ that is onto.

(b) Prove that "≤" satisfies the axioms of an order:
(i) card(A) ≤ card(A)

Lemma Suppose $B \subset A$ and $\exists f : A \to B$ which is 1-1. Then $A \sim B$.

(ii) card(A) ≤ card(B) and card(B) ≤ card(A) $\Rightarrow$ card(A) = card(B).
This is the Schroeder-Bernstein theorem.

(iii) card(A) ≤ card(B) and card(B) ≤ card(C) $\Rightarrow$ card(A) ≤ card(C).

(c) Show that
(i) $\mathbb{R} \sim (0, 1)$ Hint: Use $\mathbb{R} \sim (-\pi/2, \pi/2) \sim (0, 1)$ and $f(x) = \arctan x$.

(ii) $(0, 1) \sim [0, 1)$

(iii) $[0, 1] \sim (0, 1)$

2. Define card(A) < card(B) $\Leftrightarrow$ card(A) ≤ card(B) and card(B) $\not\leq$ card(A). In other words, there exists a 1-1 function $f : A \to B$, but none of the functions $f : A \to B$ can be onto.
Define $\mathscr{P}(A)$ = {subsets of A}, including $\varnothing$.
(i) Show that if card(A) = n, then card ($\mathscr{P}(A)$) = 2^n.

(ii) Show that card(A) < card($\mathscr{P}(A)$) for any set A.

(iii) Show that card($\mathscr{P}(\mathbb{N})$) = $\mathfrak{c}$ or in another notation $2^{\aleph_0} = \mathfrak{c}$.