1. Suppose that $\varnothing \neq B \subset A$ and there exists $f: A \to B$ which is 1-1. Prove $A \sim B$.

2. Use the construction of the sets C_n in the following cases:

(a) $B = (0, 1)$, $A = [0, 1)$, $f(x) = \frac{x}{2} + \frac{1}{2}$.

Show that $C_n = \left\{ \frac{2^n - 1}{2^n} \right\}$ and $h(a) = \begin{cases} \frac{2^{n+1}-1}{2^{n+1}} \text{ if } a = \frac{2^n - 1}{2^n} \\ a \text{ otherwise} \end{cases}$.

Then $h : [0, 1) \to (0, 1)$ is bijective.

(b) Repeat the construction for $B = (0, 1)$ $A = [0, 1]$, $f(x) = \frac{x}{3} + \frac{1}{3}$ and get a bijective function f: $[0, 1] \to (0, 1)$.

3. Schröder-Bernstein Theorem

If card(A) ≤ card(B) and card(B) ≤ card(A), then card(A) = card(B).

4. Solve Exercises 5, 6, 7, 8, 9, 10, 11 from pages 43 and 44.

Rudin 5 Construct a bounded set of real numbers with exactly 3 limit points.

Rudin 6 Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Rudin 7 Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\overline{B}_n = \bigcup_{i=1}^{n} \overline{A}_i$ for $n = 1, 2, 3, \ldots$

(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bigcup_{i=1}^{\infty} \overline{A}_i \subset \overline{B}$.

Rudin 8 Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Rudin 9 Let E° denote the set of all interior points of a set E. (E° is called the interior of E.)

(a) Prove that E° is always open.

(b) Prove that E is open $\Leftrightarrow E^\circ = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.

(e) Do E and $\overline{E}$ always have the same interiors?

(f) Do E and E° always have the same closures?

Rudin 10 Let X be an infinite set. For $p \in X$ and $q \in X$, define $d(p, q) = \begin{cases} 1 \text{ if } p \neq q \\ 0 \text{ if } p = q \end{cases}$.
Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

4.11 For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define
(a) $d_1(x, y) = (x - y)^2$.
(b) $d_2(x, y) = \sqrt{|x - y|}$
(c) $d_3(x, y) = |x^2 - y^2|$
(d) $d_4(x, y) = |x - 2y|$
(e) $d_5(x, y) = \dfrac{|x - y|}{1 + |x - y|}$
Determine, for each of these, whether it is a metric or not.

5. Consider $X = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$ and $d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$.
Show that (X, d) is a metric space.

6. Let A and B be subsets of a metric space (X, d).
(a) Show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, where $\overline{A}$ means the closure of a set A. Give an example showing that the inclusion might not be proper. (See definition 2.26 for $\overline{A}$).
(b) Show that $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.
(c) Suppose $\overline{A} \subseteq \overline{B}$. What can you say about the relationship between A and B?

7. Let (X, d_X) and (Y, d_Y) be metric spaces.
Define $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

(a) Show that $(X \times Y, d)$ is a metric space.

(b) Let $U \subset X \times Y$ be an open set.
Define $U_X = \{x \in X : \exists\, y \in Y \text{ such that } (x, y) \in U\}$ as the projection of U onto X.
1. Show that if U is open then U_X is open.
2. What can you say about U if U_X is open?
3. What can you say about U_X if U is closed?