HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

1. Check the properties of the following Cantor type sets:
(a) Divide the interval [0, 1] into 5 equal subintervals and remove the 2nd and 4th of them. Repeat this process with the remaining 3 intervals and so on...

The Cantor Set is closed, compact, perfect and is uncountable. We will check for these properties in the set described above.

Call this set C_1. Then $C_1 = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{(1/2)(5^n - 1)} \left(\frac{2k-1}{5^n}, \frac{2k}{5^n} \right)$.

(i) C_1 is closed.
(ii) C_1 is compact.

(iii) C_1 is perfect.
(iv) C_1 is uncountable.

1. (b) Again, divide the interval [0, 1] into 5 equal subintervals, but now remove the 2nd, 3rd, and 4th subintervals.
Proof:

Call this set C_2. Then $C_2 = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{5^{n-1}-1} \left(\frac{5k+1}{5^n}, \frac{5k+4}{5^n} \right)$.

The remainder of the proof is parallel to part (a) above, and the results are the same.

2. Read and understand the proof of Theorem 2.28.
(i) Theorem 2.28 Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

(ii) Let $S \subset \mathbb{R}$ be bounded above. Suppose that $\sup S = \alpha \notin S$. Show that α is a limit point of S.

3. Read the definition of convex sets from page 31.
Definition We call a set $E \subset \mathbb{R}^n$ convex if $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ whenever $\mathbf{x}, \mathbf{y} \in E$ and $0 < \lambda < 1$.
(a) Show that in $\mathbb{R}^n$, $N_r(\mathbf{x})$ and $\overline{N_r(\mathbf{x})}$ are convex.

3. (b) Show that if $S \subset \mathbb{R}^n$ is convex, then $\overline{S}$ is convex, too.

(c) Is S convex if we suppose that $\overline{S}$ is convex? No.

4. Solve exercises 17 and 18 from page 44.
Rudin 17 Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in [0, 1]? Is E compact: Is E perfect?
Is E countable?

Is E dense in [0, 1]?

Is E compact?

Is E perfect?

Rudin 18 Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

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6. Solve exercises 22 and 23 from page 45.
Rudin **22** A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^n$ is separable. **Hint**: Consider the set of points which have only rational coordinates.

Rudin **23** A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable base.
Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.

5. Solve exercises 27 and 28 from page 45.

Rudin **27** Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.
Suppose $E \subset \mathbb{R}^n$, E is uncountable, and let P be the set of all condensation points of E.

Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable.
Hint: Let $\{V_n\}$ be a countable base of $\mathbb{R}^n$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Rudin **28** Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in $\mathbb{R}^n$ has isolated points.) **Hint**: Use Exercise 27.

7. Let $S \subset \mathbb{R}$. Prove that the set of isolated points of S is at most countable.

8. Let $S \subset \mathbb{R}$ such that every $x \in S$ has a neighborhood which intersects S in at most countably many points. Show that S is at most countable.

9. Show that $(0, 1) \times (0, 1) \sim (0, 1)$.