HW 8, Practice Problems, Week #8, Assigned Thursday, 10/21/10

1. Comments regarding the proof of Theorem 2.37

Theorem 2.37

If E is an infinite subset of a compact set K, then E has a limit point in K.

(a) Rewrite the proof of Theorem 2.37 using the following hint: Suppose that E does not have limit points. Then E is closed and by Theorem 2.35 E is compact. Build now an open cover of E (not K!) which does not admit a finite subcover.

(b) Write a constructive proof of Theorem 2.37 using the following arguments: Cover K with open balls $N_1(x)$ of radius 1. Extract a finite subcover, and choose one of these balls which contains infinitely many elements of E. Name K_1 the intesection of the closure of this ball with K. Repeat the process by covering K_1 with balls of radius 1/2.

2. (a) Use Ex. 7 from Week #6 to prove the following fact:

Exercise 7: Let (X, d_X) and (Y, d_Y) be metric spaces.

Define $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. Then $(X \times Y, d)$ is a metric space.

Let $U \subset X \times Y$ be an open set.

Define $U_X = \{x \in X: \exists\, y \in Y \text{ such that } (x, y) \in U\}$ as the projection of U onto X.

Then U is open then U_X is open.

If (X, d_X) and (Y, d_Y) are metric spaces and $K \subset X$, $Q \subset Y$ are compact, then $K \times Q$ is compact in $X \times Y$.

(b) Use (a) to prove Theorem 2.40

Theorem 2.40 Every k-cell is compact.

3. Let (X, d) be a metric space.

For $\varnothing \neq K \subset X$, define the diameter of K as: $\text{diam}(K) = \sup\{d(x, y): x, y \in K\}$.

Consider a sequence $\{K_n\}$ of nonempty compact subsets of X such that :

(i) $K_{n+1} \subset K_n$, $\forall\, n \in \mathbb{N}$.

(ii) $\text{diam}(K_n) \to 0$, as $n \to \infty$.

Show that there exists a unique $x_0 \in X$ such that $\cap_{n\in\mathbb{N}} K_n = \{x_0\}$.

4. Again, let (X, d) be a metric space, $\varnothing \neq K \subset X$, and $\{G_\alpha\}$ an open cover of K. A number $\varepsilon > 0$ is called a Lebesgue number of the covering $\{G_\alpha\}$ if every $E \subset K$ with $\text{diam}(E) < \varepsilon$, is a subset of some set G_{α_0} of the covering.

Show that if K is compact, then every open cover has a Lebesgue number.

5. Let $E \subset \mathbb{R}$ be an uncountable set. Show that E has limit points.

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6. Let (X, d) be a metric space.

A nonempty $K \subset X$ is called *totally bounded* if $\forall\ \varepsilon > 0$ there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of K such that $K \subset \cup_{1 \le i \le n} N_\varepsilon(x_i)$.
$K \subset X$ is called *bounded* if K can be included in a ball of finite radius, or if $\mathrm{diam}(K) < +\infty$.

(a) Show that if K is totally bounded, then it is bounded.

(b) Does bounded imply totally bounded?
(c) Show that if K is compact, then it is totally bounded.

7. Let (X, d) be a metric space. $K \subset X$ is called nowhere dense if interior $(\overline{K}) = \varnothing$.
(a) Show that finite sets are nowhere dense in $\mathbb{R}$.
(b) Give examples of countable subsets of [0, 1] which are nowhere dense and not nowhere dense.
(c) Give an example of an uncountable nowhere dense set.
(d) Consider a sequence $\{E_n\}$ of nowhere dense subsets of [0, 1].
Show that $\cup_{n \in \mathbb{N}} E_n \neq [0, 1]$.