HW 9, Practice Problems, Week #9, Assigned Friday, 10/29/10

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.
(b) Prove the same for disjoint open sets.
(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with > is place of <. Prove that A and B are separated.
(d) Prove that every connected metric space with at leat two points is uncountable.
Hint: Use (c).

20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$)

21. Let A and B be separated subsets of some $\mathbb{R}^n$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define $\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$ for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [i.e. $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]
(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.
(b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin (A \cup B)$.
(c) Prove that every convex subset of $\mathbb{R}^n$ is connected.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Equivalent Statement:

If G_i is a dense open subset of $\mathbb{R}^n$, for i = 1, 2, 3, ..., then $\bigcap_{i=1}^{\infty} G_i \neq \varnothing$ (in fact it is dense in $\mathbb{R}^n$).

(*This is a special case of Baire's theorem; see Exercise 22, chapter 3 for the general case.*)