Real Analysis Comprehensive Exam, Dec 2008

1. (a) Let A be the set of all sequences whose elements are the digits 0 and 1. Prove that the set A is uncountable.

(b) For a set A, the family of all subsets of A is said to be the power set of A and is denoted by $\mathscr{P}(A)$. Prove that $\mathscr{P}(\mathbb{N})$ is uncountable.

2. Prove that every uncountable set in $\mathbb{R}^2$ has at least a limit point.

3. Suppose that f is a continuous mapping of a metric space X onto a compact metric space Y. Is X necessarily compact? If so, prove it; otherwise, provide a counterexample.

4. Let $\{I_n\}$ be a decreasing sequence of closed intervals in $\mathbb{R}$, i.e. $[a_{n+1}, b_{n+1}] = I_{n+1} \subseteq I_n = [a_n, b_n]$ for $n = 1, 2, 3, \ldots$. Prove that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$. Furthermore, if $\lim_{n\to\infty}(b_n - a_n) = 0$, prove that $\bigcap_{n=1}^{\infty} I_n = \{x_0\}$ for some $x_0 \in \mathbb{R}$.

5. Let s_n be a bounded sequence of real numbers such that $2s_n \leq s_{n-1} + s_{n+1}$. Show that $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$.

6. If $\{s_n\}$ is a sequence of real numbers and if $\sigma_n = \dfrac{s_1 + s_2 + \cdots + s_n}{n}$, $n \in \mathbb{N}$, prove that $\limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n$ and $\liminf_{n\to\infty} \sigma_n \geq \liminf_{n\to\infty} s_n$.

7. (a) Let f and g be continuous real-valued functions on $\mathbb{R}$.
Let A be the set of all $x \in \mathbb{R} \ni f(x) < g(x)$.
Use the ε-δ definition of continuity to prove that A is open.

(b) Let f be an increasing function defined on (a, b) and let $c \in (a, b)$. Suppose that f is continuous at c. Prove that $\sup\{f(x) : a < x < c\} = f(c)$.

8. (a) Suppose that f is a continuous function with $f(x) > 0$ for all $x \in \mathbb{R}$, and that $\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to+\infty} f(x)$. Prove that there is some number $x_0 \ni f(x_0) \geq f(x)$ for all x.

(b) If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, prove that $\lim_{x\to b-} f(x)$ exists.

9. (a) Prove that if $|f|$ is differentiable at a and f is continuous at a, then f is also differentiable at a.

(b) Let f be monotone increasing on $[a, b]$ and suppose that f is discontinuous at $c \in (a, b)$. Show that $F(x) = \int_a^x f$ is not differentiable at $x = c$.

10. Define a function f on $\mathbb{R}$ as follows: $f(x) = \begin{cases} x^2 \cos\frac{1}{x} : x \neq 0 \\ 0 : x = 0 \end{cases}$. Prove that f is differentiable for all values of $x \in \mathbb{R}$, but f' is not continuous on $\mathbb{R}$.

11. Let $\{r_n\}$ be a listing of the rational numbers in [0, 1]. Define $f(x) = \begin{cases} \frac{1}{n} : x = r_n \\ 0 : x \notin Q \end{cases}$.

Determine whether $f \in \mathscr{R}[0, 1]$. Justify your answer.

12. Let f be a bounded function on [−1, 1] and let $\alpha(x) = \begin{cases} 0 : x \leq 0 \\ 1 : x > 0 \end{cases}$. Prove that $f \in \mathscr{R}(\alpha)$ $\Leftrightarrow f(0+) = f(0)$. If $f \in \mathscr{R}(\alpha)$, compute $\int_{-1}^{1} f(x)d\alpha(x)$.

13. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. For $n \in \mathbb{N}$ define $f_n(x) = f(x + \frac{1}{n})$. Prove that $\{f_n\}$ converges uniformly on $\mathbb{R}$.

(b) For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, define $f_n(x) = nxe^{-nx^2}$. Show that $\{f_n\}$ converges pointwise on $[0, \infty)$ and find the pointwise limit function f. Determine whether the convergence is uniform on $[0, \infty)$. Justify your answer.

14. (a) Suppose that f_n converges uniformly on $[a, b]$ to f and each f_n is bounded on $[a, b]$, say $\sup_{x \in [a, b]} |f_n(x)| \leq B_n$. Prove that there is a uniform bound B such that for every $n \in \mathbb{N}$ and every $x \in [a, b]$, $|f_n(x)| \leq B$.

(b) Assume that $f_n \to f$ and $g_n \to g$ uniformly on an interval $I \subset \mathbb{R}$. Also assume that each f_n and g_n is bounded on I. Prove that $f_n g_n \to fg$ uniformly on I.

15. Assume that the sequence $\{f_n\}$ of functions converges uniformly on $[a, b]$ to f and that each f_n is Riemann integrable on $[a, b]$. Prove that f is integrable on $[a, b]$ and $\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$.

16. (a) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in some interval $(-R, R)$ and that $f(x) = 0$ for all x in $(-R, R)$. Prove that $a_n = 0$ for all $n = 0, 1, 2, \ldots$.

(b) Prove that the series $\sum_{n=0}^{\infty} \frac{nx}{1 + n^4 x^2}$ converges uniformly on $[a, \infty)$ for $a > 0$ but does not converge uniformly on $(0, \infty)$.