Real Analysis Comprehensive Exam, May 2008

1. (a) Prove that every uncountable subset of $\mathbb{R}$ has a limit point.

(b) Prove that any connected metric space with at least two distinct points is uncountable.

2. Prove that the Cantor set is uncountable and perfect.

3. Prove that the compact subsets of metric spaces are closed and bounded. Is the converse true? Justify your answer.

4. Let E be a connected subset of a metric space X and let $A \subset X$ be such that $E \subseteq A \subseteq \mathrm{cl}(E)$. Prove that A is connected.

5. Show that every bounded monotonic sequence of real numbers is Cauchy. (Do not use the monotone convergence theorem.)

6. (a) Let $\{a_n\}$ be a sequence of real numbers. State the definition of $\limsup_{n\to\infty} a_n$.

(b) For any sequence $\{a_n\}$ of real numbers, let $b_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$. Prove that $\limsup_{n\to\infty} b_n \le \limsup_{n\to\infty} a_n$.

7. Let $\Sigma_{n=1}^{\infty} a_n$ be a series of real numbes.

(a) Assume that $\Sigma_{n=1}^{\infty} |a_n|$ converges. Prove that $\Sigma_{n=1}^{\infty} a_n$ converges. Give an example that shows that the converse is not generally true. Justify your conclusions.

(b) Construct an example of a series $\Sigma_{n=1}^{\infty} (-1)^n a_n$, where $a_m > 0$ for all n = 1, 2, ..., that does not converge, even though $a_n \to 0$.

8. Suppose that $f:(a, b) \to \mathbb{R}$ has the intermediate value property and that f takes on each value only one. Prove that f is continous on (a, b).

9. (a) Assume that f is continuous on $[a, b]$. Use the definition to prove that f is uniformly continuous on $[a, b]$.

(b) Is $f(x) = x\sin x$ uniformly continuous on $[0, \infty)$? Justify your answer.

10. (a) Suppose that $f'(x) \ge c > 0$ for all $x \in (0, \infty)$. Prove that $\lim_{x\to\infty} = \infty$.

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

11. Suppose f is differentiable at every point on (a, b). Prove that f' is increasing on (a, b), then f' is continuous on (a, b).

12. Let $f(x)$ be defined on $[0, a]$ such that

$$f(x) = \begin{cases} \frac{1}{n} : x = \frac{1}{2^n},\ n = 1,\ 2,\ldots, \\ \text{and } j \text{ is an odd integer such that } 1 \le j \le 2^n - 1\ . \\ 0 : \text{otherwise} \end{cases}$$

Use the definition to prove that $f(x)$ is Riemann integrable on [0, 1]. Find $\int_0^1 f(x)dx$.

13. Assume that f is bounded on $[a, b]$, f is continuous at x, $a < s < b$, and $\alpha(x) = c \bullet I(x - s)$, where c is a constant and $f(x)$ is the unit step function defined as $I(x) = \begin{cases} 0 : x \le 0 \\ 1 : x > 0 \end{cases}$. Prove that $f \in R(\alpha)$. What is the value of $\int_a^b f d\alpha$?

14. Let $\{f_n\}$ be a sequence of continuous nondecreasing functions defined on an interval $[a, b]$. Suppose f_n converges to f pointwise on $[a, b]$. Prove that if f is continuous on $[a, b]$, then the convergence is uniform. Note that here each function $f_n(x)$, not the sequence $\{f_n\}$, is assumed monotonic.

15. Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$. Prove that $\{f_n\}$ converges uniformly on f on $[a, b]$ if and only if $f_n(x_n) \rightarrow f(x_0)$ whenever $\{x_n\}$ is a sequence of points in $[a, b]$ that converges to a point $x_0 \in [a, b]$.

16. For what values of $x \ge 0$ does the series $\sum_{n=1}^{\infty} \frac{1}{k + k^2 x}$ converge? Is the convergence uniform on the set where the series converge? Justify your answer.