Real Analysis Comprehensive Exam, May 2009

1. Given a set $S \subset \mathbb{R}^n$ with the property that for every $x \in S$, there exists a ball B centered at x such that $B \cap S$ is countable, prove that S is countable.

2. Let $C \subset [0, 1]$ be the Cantor set. Prove that C is perfect.

3. Prove that any nonempty connected subset E of $\mathbb{R}$ is an interval.

4. Let $\{a_n\}$ be a sequence of real numbers bounded from below, and let $A = \{p$: there is a subsequence of $\{a_n\}$ converging to $p\}$. Suppose $A \neq \varnothing$. Define $a = \inf A$. Prove that $a \in A$ and that, for each $\varepsilon > 0$, there is N such that for all $n \geq N$, $a - \varepsilon < a_n$, and there are infinitely many m such that $a_m < a + \varepsilon$.

5. Show that the sequence $s_k = \sum_{n=1}^{k} \frac{1}{\sqrt{k^2+n}}$, $k = 1, 2, 3, \ldots$, converges. Find the limit.

6. Suppose that $f:[a, b] \to \mathbb{R}$ is continuous and define $g(x) = \max\{f(t) : a \leq t \leq x\}$, $a \leq x \leq b$. Prove that g is continuous on $[a, b]$.

7. Assume that $f:(a, b) \to \mathbb{R}$ is differentiable and f' is bounded. Prove that $\lim_{x \to b-} f(x)$ exists.

8. Assume that $f: \mathbb{R} \to \mathbb{R}$ is continuous with the property that for every open subset G of $\mathbb{R}$, $f(G)$ is open. Prove that f is monotonic on $\mathbb{R}$.

9. Suppose $f:(a, b) \to \mathbb{R}$ is differentiable and $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b). Give an example of a function $f:(0, 1) \to \mathbb{R}$ that is differentiable and uniformly continuous on (0 1) but such that f' is unbounded. Justify your answer.

10. Suppose f, g, and h are defined on (a, b) and $a < x_0 < b$. Assume that f and h are differentiable at x_0, $f(x_0) = h(x_0)$, and $f(x) \leq g(x) \leq h(x)$ for all x in a neighborhood of x_0. Prove that g is differentiable at x_0 and $f'(x_0) = g'(x_0) = h'(x_0)$.

11. If a function f is twice differentiable on $(0, \infty)$ and $f''(x) \geq c > 0$ for all x, prove that f is not bounded from above.

12. (a) Assume that f is integrable on $[a, b]$ and define $F(x) = \int_a^x f(t)dt$, $a \leq x \leq b$. Prove that F is continuous on $[a, b]$.
(b) Assume that f is continuous on $[a, b]$. Prove that $F(x) = \int_a^x f(t)dt$ is differentiable on $[a, b]$ and that $F'(x) = f(x)$ for every $x \in [a, b]$.

13. Suppose that f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Prove that $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

14. Suppose that f is continuous on $[0, 1]$. Define $g_n(x) = f(x^n)$ for $n = 1, 2, \ldots$. Prove that $\{\int_0^1 g_n(x)dx\}$ converges to $f(0)$.

15. Suppose that the sequence $\{f_k(x)\}$ on the interval $[0, 1]$ satisfies $|f_k(s) - f_k(t)| \leq |s - t|$ for all $s, t \in [0, 1]$. Assume also that $\{f_k(x)\}$ converges pointwise to $f(x)$ on $[0, 1]$. Prove that $\{f_k(x)\}$ converges uniformly on $[0, 1]$.

16. Does the series $\sum_{k=0}^{\infty} \frac{x^2}{(1+x^2)^k}$ converge pointwise on $\mathbb{R}$? Does it converge uniformly on $[a, \infty)$, $a > 0$? Does it converge uniformly on $\mathbb{R}$? Justify your answer.