Do 4 of the following problems:

1. (a) What fields can be a homomorphic image of $\mathbb{Q}[x]/(x^4 - 4)$?
(b) What fields can be a homomorphic image of $\mathbb{R}[x]/(x^4 - 4)$?
(c) What fields can be a homomorphic image of $\mathbb{C}[x]/(x^4 - 4)$?

2. Let R be a PID and let I and J be ideals of R, with $I = (a)$ and $J = (b)$.
Show that $IJ = I \cap J \Leftrightarrow ab = \text{lcm}\,(a, b)$.

3. Let K be a splitting field for $x^8 - 4 \in \mathbb{Q}[x]$.
(a) Find $[K{:}\mathbb{Q}]$.
(b) Show that $\text{Gal}(K/\mathbb{Q})$ is not cyclic.

4. (a) Find a basis for $\mathbb{Q}[x]/(x^2 + x + 1)$ as a vector space over $\mathbb{Q}$.
(b) Find a multiplicative inverse for $x + (x^2 + x + 1) \in \mathbb{Q}[x]/(x^2 + x + 1)$.
(c) Show that $x + (x^3 + x^2 + x) \in \mathbb{Q}[x]/(x^3 + x^2 + x)$ does not have a multiplicative inverse.

5. For R a commutative ring with 1, define $N(R) = \{x \in \text{R}: x^n = 0 \text{ for some } n \in \mathbb{N}\}$.
(a) Show that $N(R)$ is an ideal of R.
(b) Show that $N(R) \subseteq P$ for every prime ideal P of R.
(c) Show that $R/N(R)$ has no non-zero nilpotents.
(d) For $R = \mathbb{Q}[x]/(x^3 + 2x^2)$, find $N(R)$.