Comprehensive Exam, Algebra December 2008

Part I: Group Theory (Do 4 of the following 5 problems)

1. Let G be a group of order 3 • 13 • 19 = 741. Show that its 13-Sylow and 19-Sylow subgroups are normal in G.

2. Note: $135 = 3^3 \bullet 5$.
(a) Up to isomorphism, find all abelian groups of order 135.
(b) For each isomorphism class found in part (a), how many elements are there or order 3? of order 15?
(c) For each isomorphism class found in part (a), how many subgroups are there of order 3? or order 15?

3. Let G be an abelian group and let $H = \{g \in G \mid g \text{ has finite order}\}$.
(a) Show that H is a subgroup of G.
(b) Show that G/H has exactly one element of finite order.

4. Let m and n be positive integers.
(a) Show that there is a non-trivial homomorphism from $\mathbb{Z}_m$ to $\mathbb{Z}_n \Leftrightarrow m$ and n are not relatively prime.
(b) Give an example (or explain why none exists) of a
(i) homomorphism from S_3 onto $\mathbb{Z}_3$
(ii) homomorphism from S_3 onto $\mathbb{Z}_2$

5. Let G be a group and let H be a subgroup of G.
(a) Prove that $H \lhd G \Leftrightarrow H$ is a union of conjugacy clases.
(b) Determine all conjugacy classes in S_5 and determine the size of each class.
(c) Use parts (a) and (b) to explain why S_5 has exactly three normal subgroups, S_5, A_5 and $\{(1)\}$.
(d) Let $\sigma = (123) \in S_5$. Determine the centralizer of σ in S_5.

Part II: Ring Theory (Do 4 of the following 5 problems)
Do 4 of the following problems:

1. Let K be a splitting field for $x^8 - 2$ over $\mathbb{Q}$.
(a) Find $[K:\mathbb{Q}]$.
(b) Show that the Galois group for K over $\mathbb{Q}$ is not cyclic.
(c) Let σ be the element in the Galois group for which $\sigma(\sqrt[8]{2}) = i\sqrt[8]{2}$ and $\sigma(i) = -i$. Find $[L:\mathbb{Q}]$ where L is the fixed field of σ.

2. Let R be the ring of continuous functions on the closed interval $[0, 1]$.
Let $M = \{f \in R: f(1/2) = 0\}$.
(a) Prove that M is an ideal in R.
(b) Prove R/M is isomorphic to $\mathbb{R}$.
(c) Prove that M is a maximal ideal of R.

3. (a) Let F be a field and let E be an extension field of F. Suppose $[E:F] = p$ for some prime p. Let $a \in E$. Prove $F(a) = F$ or $F(a) = E$.
(b) Suppose $[F(a):F] = 5$. Determine, with explanation, $[F(a^3):F]$.
(c) Let p be a prime and let m and n be natural numbers. Prove that if $\mathbb{F}_{p^n}$ is a subfield of $\mathbb{F}_{p^m}$, then $n|m$. (Note: $\mathbb{F}_{p^n}$ denotes the finite field with p^n elements.)

4. A commutative ring R with 1 is called a local ring in case R has a unique maximal ideal.
(a) Show that if R is local, then every homomorphic image of R is local.
(b) Find a homomorphic image of $\mathbb{Z}$ that is a local ring, but not a field.

5. Let R be a commutative ring with unity and let I be an ideal in R. For each $a \in R$, define $I_a = \{r \in R: ar \in I\}$.
(a) Prove that I_a is an ideal of R containing I.
(b) Prove that if I is a prime ideal, then $I_a = I$ or $I_a = R$.
(c) Let $R = \mathbb{Q}[x]$, let $I = (x^4 - 1)$ and let $a = x^2 + 1$. Find an element $b \in R \ni I_a = (b)$.