Do 4 of the following problems:

1. Let K be a splitting field for $x^8 - 2$ over $\mathbb{Q}$.
(a) Find $[K:\mathbb{Q}]$.
(b) Show that the Galois group for K over $\mathbb{Q}$ is not cyclic.
(c) Let σ be the element in the Galois group for which $\sigma(\sqrt[8]{2}) = i\sqrt[8]{2}$ and $\sigma(i) = -i$. Find $[L:\mathbb{Q}]$ where L is the fixed field of σ.

2. Let R be the ring of continuous functions on the closed interval $[0, 1]$.
Let $M = \{f \in R: f(1/2) = 0\}$.
(a) Prove that M is an ideal in R.
(b) Prove R/M is isomorphic to $\mathbb{R}$.
(c) Prove that M is a maximal ideal of R.

3. (a) Let F be a field and let E be an extension field of F. Suppose $[E:F] = p$ for some prime p. Let $a \in E$. Prove $F(a) = F$ or $F(a) = E$.
(b) Suppose $[F(a):F] = 5$. Determine, with explanation, $[F(a^3):F]$.
(c) Let p be a prime and let m and n be natural numbers. Prove that if $\mathbb{F}_{p^n}$ is a subfield of $\mathbb{F}_{p^m}$, then $n|m$. (Note: $\mathbb{F}_{p^n}$ denotes the finite field with p^n elements.)

4. A commutative ring R with 1 is called a local ring in case R has a unique maximal ideal.
(a) Show that if R is local, then every homomorphic image of R is local.
(b) Find a homomorphic image of $\mathbb{Z}$ that is a local ring, but not a field.

5. Let R be a commutative ring with unity and let I be an ideal in R. For each $a \in R$, define $I_a = \{r \in R: ar \in I\}$.
(a) Prove that I_a is an ideal of R containing I.
(b) Prove that if I is a prime ideal, then $I_a = I$ or $I_a = R$.
(c) Let $R = \mathbb{Q}[x]$, let $I = (x^4 - 1)$ and let $a = x^2 + 1$. Find an element $b \in R \ni I_a = (b)$.