Do 4 of the following problems:

1. Let R be a ring with unity.
(a) Let I be an ideal of R. Suppose that every nonzero coset in R/J contains a unit of R. Prove that the only ideals of R/I are the trivial ideals I and R/I.
(b) Suppose R is a commutative ring with unity. Prove that R is a field $\Leftrightarrow$ the only ideals of R are the trivial ideals $\{0\}$ and R.

2. Let S be a Euclidean domain.
(a) Prove that S is a ring with unity.
(b) Prove that S is a principal ideal domain.
(c) Let $I = (a)$ be an ideal in a principal ideal domain.
Prove that I is a maximal ideal $\Leftrightarrow$ a is irreducible.

3. Prove or disprove each of the following:
(a) $\mathbb{Z}_5[x]/(x^2 + 2)$ is isomorphic to $\mathbb{Z}_5[x]/(x^2 - 2)$.
(b) $\mathbb{Q}_5[x]/(x^2 + 2)$ is isomorphic to $\mathbb{Q}_5[x]/(x^2 - 2)$.

4. Let E be an extension field of a field F.
(a) Let $a \in E$. Prove a is algebraic over $F \Leftrightarrow [F(a):F] < \infty$.
(b) Let $A = \{a \in E\text{: } a \text{ is algebraic over } F\}$. Prove that A is a subfield of E containing F.

5. Let $p(x) = x^4 - 5 \in \mathbb{Q}[x]$ and let E denote the splitting field for $p(x)$ over $\mathbb{Q}$.
(a) Show that $E = \mathbb{Q}(i, \sqrt[4]{5})$.
(b) Describe all of the automorphisms of E over $\mathbb{Q}$ and explain how you know that they are all automorphisms (without directly verifying that they satisfy the definition of an automorphism).
(c) Let τ be the automorphism which sends i to i and $\sqrt[4]{5}$ to $i\sqrt[4]{5}$ and let σ be the automorphism which sends i to $-i$ and $\sqrt[4]{5}$ to $i\sqrt[4]{5}$. Determine the permutation of the roots of $p(x)$ which corresponds to each of these automorphisms. Use this information to show $\text{Gal}(E/\mathbb{Q}) \cong D_4$ (where D_4 denotes the dihedral group with 8 elements).
(d) Determine, with explanation, the fixed field of the automorphism σ (where σ is an defined in part (c)).