Comprehensive Exam, Algebra, 2009

Part I: Group Theory (Do 4 of the following 5 problems)

1. (a) Let G be a cyclic group of order n. Prove G has a unique subgroup of order k for each positive integer k dividing n.
(b) Up to isomorphism, find all abelian groups of order $50 = 5^2 \bullet 2$. For each isomophism class, determine the number of subgroups of order 5 in each.

2. (a) Suppose that H is a cyclic subgroup of G and that H is a normal subgroup of G. Prove that every subgroup of H is normal in G.
(b) Prove that if $[G:H] = 2$, then H is a normal subgroup of G.

3. (a) Let G be a group and H a subgroup of G. Show that $H \lhd G$ if and only if H is a union of conjugacy classes.
(b) Determine all conjugacy classes in S_5 and determine the size of each class.
(c) Use parts (a) and (b) to explain why S_5 has exactly three normal subgroups, S_5, A_5 and $\{(1)\}$.
(d) Determine the maximal order of an element of S_5 and show that if σ is an element of maximal order in S_5, then $C_G(\sigma) = \langle\sigma\rangle$.

4. (a) Let $\mathbb{Z}$ be the center of the group G. Prove that if $G/\mathbb{Z}$ is cyclic, then G is abelian.
(b) Let G be a nonabelian group.
Prove that if $|G| = p^3$ for some prime p, then $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

5. (a) Let G be a group of order 12.
i. Prove that G is not simple
ii. Suppose that G has more than one 3-Sylow subgroup. Prove that $N_G(K) = K$ for every 3-Sylow subgroup K of G.
(b) Let G be a finite group and let p be a prime dividing $|G|$. Suppose P_1 and P_2 are p-Sylow subgroups with $P_1 \neq P_2$. Show $N(P_1) \neq N(P_2)$.

Part II: Ring and Field Theory (Do 4 of the following 5 problems)

1. Let K be a splitting field for $x^8 - 2$ over $\mathbb{Q}$.
(a) Find, with explanation, $[K:\mathbb{Q}]$.
(b) Show that the Galois group for K over $\mathbb{Q}$ is not cyclic.
(c) Let σ be the element in the Galois group for which $\sigma(\sqrt[8]{2}) = i\sqrt[8]{2}$ and $\sigma(i) = -i$. Find, with explanation, $[L:\mathbb{Q}]$ where L is the fixed field of σ.

2. Let R be a commutative ring. Let I be an ideal in R. Define $R(I) = \{x \in R: x^n \in I$ for some $n \in \mathbb{Z}\}$.
(a) Prove that $R(I)$ is an ideal in R.
(b) Prove that if P is a prime ideal with $I \subseteq P$, then $R(I) \subseteq P$.
(c) Prove that if $I = R(I)$, then R/I contains no nonzero nilpotent elements.

3. **(a)** Let F be a field and let E be an extension field of F. Prove that the elements of E which are algebraic over F form a subfield of E.
(b) Let $F = \mathbb{Q}$ and let $E = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots)$.
i. Prove that E is an algebraic extension of F.
ii. Prove $[E:F] = \infty$.

4. Determine each of the following (with explanation).
(a) All ideals in $\mathbb{Q}[x]/(x^4 - x^2 - 12)$.
(b) All fields which are homomorphic images of $\mathbb{Z}_{20}$.
(c) Describe all subfields of $\mathbb{C}$ which are homomorphic images of $\mathbb{Q}[x]/(x^4 + x^2)$?

5. Let R be a principal ideal domain.
(a) Let $I = (a)$ be a nonzero ideal in R. Prove that I is a prime ideal if and only if a is irreducible.
(b) Prove every nonzero prime ideal in R is a maximal ideal.
(c) Suppose $I_0 \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ where each I_n is an ideal in R. Prove there exists some natural number N such that $I_j = I_k$ for all $j, k \geq N$. (Hint: for a first step, show $\bigcup_{n\geq 0} I_n$ is an ideal in R.)