

**Part I: Group Theory** (Do 4 of the following 5 problems)

1. Let  $G$  be a group of order  $3 \cdot 13 \cdot 19 = 741$ . Show that its 13-Sylow and 19-Sylow subgroups are normal in  $G$ .
2. Note:  $135 = 3^3 \cdot 5$ .
  - (a) Up to isomorphism, find all abelian groups of order 135.
  - (b) For each isomorphism class found in part (a), how many elements are there of order 3? of order 15?
  - (c) For each isomorphism class found in part (a), how many subgroups are there of order 3? or order 15?
3. Let  $G$  be an abelian group and let  $H = \{g \in G \mid g \text{ has finite order}\}$ .
  - (a) Show that  $H$  is a subgroup of  $G$ .
  - (b) Show that  $G/H$  has exactly one element of finite order.
4. Let  $m$  and  $n$  be positive integers.
  - (a) Show that there is a non-trivial homomorphism from  $\mathbb{Z}_m$  to  $\mathbb{Z}_n \Leftrightarrow m$  and  $n$  are not relatively prime.
  - (b) Give an example (or explain why none exists) of a
    - (i) homomorphism from  $S_3$  onto  $\mathbb{Z}_3$
    - (ii) homomorphism from  $S_3$  onto  $\mathbb{Z}_2$
5. Let  $G$  be a group and let  $H$  be a subgroup of  $G$ .
  - (a) Prove that  $H \triangleleft G \Leftrightarrow H$  is a union of conjugacy classes.
  - (b) Determine all conjugacy classes in  $S_5$  and determine the size of each class.
  - (c) Use parts (a) and (b) to explain why  $S_5$  has exactly three normal subgroups,  $S_5$ ,  $A_5$  and  $\{(1)\}$ .
  - (d) Let  $\sigma = (123) \in S_5$ . Determine the centralizer of  $\sigma$  in  $S_5$ .

**Part II: Ring Theory** (Do 4 of the following 5 problems)

Do 4 of the following problems:

1. Let  $K$  be a splitting field for  $x^8 - 2$  over  $\mathbb{Q}$ .
  - (a) Find  $[K:\mathbb{Q}]$ .
  - (b) Show that the Galois group for  $K$  over  $\mathbb{Q}$  is not cyclic.
  - (c) Let  $\sigma$  be the element in the Galois group for which  $\sigma(\sqrt[8]{2}) = i\sqrt[8]{2}$  and  $\sigma(i) = -i$ . Find  $[L:\mathbb{Q}]$  where  $L$  is the fixed field of  $\sigma$ .
  
2. Let  $R$  be the ring of continuous functions on the closed interval  $[0, 1]$ . Let  $M = \{f \in R : f(1/2) = 0\}$ .
  - (a) Prove that  $M$  is an ideal in  $R$ .
  - (b) Prove  $R/M$  is isomorphic to  $\mathbb{R}$ .
  - (c) Prove that  $M$  is a maximal ideal of  $R$ .
  
3. (a) Let  $F$  be a field and let  $E$  be an extension field of  $F$ . Suppose  $[E:F] = p$  for some prime  $p$ . Let  $a \in E$ . Prove  $F(a) = F$  or  $F(a) = E$ .
  - (b) Suppose  $[F(a):F] = 5$ . Determine, with explanation,  $[F(a^3):F]$ .
  - (c) Let  $p$  be a prime and let  $m$  and  $n$  be natural numbers. Prove that if  $\mathbb{F}_{p^m}$  is a subfield of  $\mathbb{F}_{p^n}$ , then  $n|m$ . (Note:  $\mathbb{F}_{p^n}$  denotes the finite field with  $p^n$  elements.)
  
4. A commutative ring  $R$  with 1 is called a local ring in case  $R$  has a unique maximal ideal.
  - (a) Show that if  $R$  is local, then every homomorphic image of  $R$  is local.
  - (b) Find a homomorphic image of  $\mathbb{Z}$  that is a local ring, but not a field.
  
5. Let  $R$  be a commutative ring with unity and let  $I$  be an ideal in  $R$ . For each  $a \in R$ , define  $I_a = \{r \in R : ar \in I\}$ .
  - (a) Prove that  $I_a$  is an ideal of  $R$  containing  $I$ .
  - (b) Prove that if  $I$  is a prime ideal, then  $I_a = I$  or  $I_a = R$ .
  - (c) Let  $R = \mathbb{Q}[x]$ , let  $I = (x^4 - 1)$  and let  $a = x^2 + 1$ . Find an element  $b \in R \ni I_a = (b)$ .