Comprehensive Exam, Algebra, 2010

Part I Group Theory (Do 4 of the following 5 problems)

1. Let G be an abelian group of order $5^3 \cdot 3 = 375$.
(a) Determine all possible isomorphism classes of G.
(b) For each isomorphism class, how many elements are there of order 3?
(c) For each isomorphism class, how many elements are there of order 5?
(d) If there are 48 elements of order 15, what is the isomorphism class of G?

2. (a) In S_7, find two elements σ and τ of order 6 that are not conjugate.
(b) How many conjugates does each have?
(c) Find the centralizer of σ and the centralizer of τ.

3. Let G be a group such that $|G| = p^n$ for some prime p.
(a) Prove that $Z(G) \neq \langle e \rangle$.
(b) Prove that G has a normal subgroup of order p^{n-1}.
(c) Prove that if $N \neq \langle e \rangle$ is a normal subgroup of G, then $N \cap Z(G) \neq \langle e \rangle$.

4. Let G be a group and $H = \{g^2 | g \in G\}$.
(a) Prove that if $H \leq G$, then $H \lhd G$.
(b) Prove that if G is abelian, then $H \leq G$.
(c) Prove that if G is finite, and $H \leq G$, then $|G/H| = 2^n$.

5. Let G be a group such that $|G| = p^2q$ where p and q are primes, $p \neq q$.
(a) Prove that G is not simple.
(b) Prove that if $p = 7$ and $q = 13$, then G is abelian.

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Part II Ring and Field Theory (Do 4 of the following 5 problems)

1. Let R be a commutative ring with unity and let I be an ideal of R.
(a) Prove that if I is a maximal ideal, then I is a prime ideal.
(b) Prove that R is a field if and only if the only ideals of R are $\{0\}$ and R.
(c) Suppose that R satisfies the property that, for each $a \in R$, there exists a natural number $n > 1$ such that $a^n = a$. Prove that every prime ideal in R is a maximal ideal in R.

2. Let $p(x) = (x^2 - 9)(x^2 + 4) \in \mathbb{Q}[x]$. Let $I = (p(x))$.
(a) Determine, with proof, all ideals in $\mathbb{Q}[x]$ which contain I.
(b) For which of the ideals, J, from part (a) is $\mathbb{Q}[x]/J$ a field? (Explain.)
(c) Determine all ideals in $\mathbb{Q}(i)[x]/I$.

3. Let p be a prime and let F be an extension of $\mathbb{Z}_p$ such that $[F:\mathbb{Z}_p] = n$.
(a) Prove $|F| = p^n$.
(b) Prove F is a normal extension of $\mathbb{Z}_p$.
(c) Define $\sigma: F \to F$ by $\sigma(a) = a^p$ for each $a \in F$. Prove that $\mathrm{Gal}(F/\mathbb{Z}_p) = \langle\sigma\rangle$.

4. Let $p(x) = x^4 - 3 \in \mathbb{Q}[x]$ and let E denote the splitting field for $p(x)$ over $\mathbb{Q}$.
(a) Show that $E = \mathbb{Q}(\sqrt[4]{3}, i)$.
(b) Describe all of the automorphisms of E over $\mathbb{Q}$ and explain how you know that they are all automorphisms wthout directly verifying that they satisfy the definition of an automorphism.
(c) Let τ be the automorphism such that $\tau(i) = i$ and $\tau(\sqrt[4]{3}) = i\sqrt[4]{3}$ and let σ be the automorphism such that $\sigma(i) = -i$ and $\sigma(\sqrt[4]{3}) = i\sqrt[4]{3}$. Determine the permutation of the roots of $p(x)$ which corresponds to each of these automorphisms.
(d) Prove $\mathrm{Gal}(E/\mathbb{Q}) \cong D_8$ (where D_8 denotes the dihedral group with 8 elements).
(e) Determine, with explanation, the fixed field of the automorphism σ.

5. Let E be the set of all algebraic elements over F.
(a) Prove that E is a field.
(b) Prove that if $F = \mathbb{Q}$, then $[E:F] = \infty$.