1. From Final Exam, May 2010 #6.
Let $f(x) = x^4 - 5 \in \mathbb{Q}[x]$. The roots of $f(x)$ are $\pm\sqrt[4]{5}$ and $\pm i\sqrt[4]{5}$ and the splitting field for $f(x)$ is $E = \mathbb{Q}(\sqrt[4]{5}, i)$.
(a) Describe all of the automorphisms of E over $\mathbb{Q}$ and explain how you know that they are all automorphisms without directly verifying that they satisfy the definition of an automorphism.
(b) Let τ be the automorphism such that $\tau(i) = i$ and $(\sqrt[4]{5}) = -\sqrt[4]{5}$ and let σ be the automorphism such that $\sigma(i) = -i$ and $\sigma(\sqrt[4]{5}) = i\sqrt[4]{5}$. Determine the permutation of the roots of $p(x)$ which corresponds to each of these automorphisms.

2. From Final Exam, May 2010 #7.
Let $p(x) = x^3 - 5 \in \mathbb{Q}[x]$.
(a) Let K be the splitting field for $p(x)$ over $\mathbb{Q}$. Prove that $K = \mathbb{Q}(\sqrt[3]{5}, i\sqrt{3})$.
(b) Prove $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$.
(c) Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[3]{5}, i\sqrt{3})$. Explain how you know that you have found all of them.

3. From Comp, May 2010 #4.
Let $p(x) = x^4 - 3 \in \mathbb{Q}[x]$ and let E denote the splitting field for $p(x)$ over $\mathbb{Q}$.
(a) *Show that $E = \mathbb{Q}(\sqrt[4]{3}, i)$.*
(b) *Describe all of the automorphisms of E over $\mathbb{Q}$ and explain how you know that they are all automorphisms wthout directly verifying that they satisfy the definition of an automorphism.*
(c) *Let τ be the automorphism such that $\tau(i) = i$ and $\tau(\sqrt[4]{3}) = i\sqrt[4]{3}$ and let σ be the automorphism such that $\sigma(i) = -i$ and $\sigma(\sqrt[4]{3}) = i\sqrt[4]{3}$. Determine the permutation of the roots of $p(x)$ which corresponds to each of these automorphisms.*
(d) *Prove $\mathrm{Gal}(E/\mathbb{Q}) \cong D_8$ (where D_8 denotes the dihedral group with 8 elements).*
(e) *Determine, with explanation, the fixed field of the automorphism σ.*

4. From Comp, May 2009 #1.
Let K be a splitting field for $x^8 - 2$ over $\mathbb{Q}$.
(a) Find, with explanation $[K:\mathbb{Q}]$.
(b) Show that the Galois group for K over $\mathbb{Q}$ is not cyclic.
(c) Let σ be the element in the Galois group for which $\sigma(\sqrt[8]{2}) = i\sqrt[8]{2}$ and $\sigma(i) = -i$. Find, with explanation $[L:\mathbb{Q}]$ where L is the fixed field of σ.

5. From Comp, Dec 2008 #1.
Let K be a splitting field for $x^8 - 2$ over $\mathbb{Q}$.
(a) Find $[K:\mathbb{Q}]$.
(b) Show that the Galois group for K over $\mathbb{Q}$ is not cyclic.
(c) Let σ be the element in the Galois group for which $\sigma(\sqrt[8]{2}) = i\sqrt[8]{2}$ and $\sigma(i) = -i$. Find $[L:\mathbb{Q}]$ where L is the fixed field of σ.

6. From Final Exam, May 2008 #3.
Let $g(x) = x^4 - 3 \in Q[x]$. The roots of this polynomial are $a_1 = \sqrt[4]{3}$, $a_2 = i\sqrt[4]{3}$, $a_3 = -\sqrt[4]{3}$ and $a_4 = -i\sqrt[4]{3}$, and the splitting field for $g(x)$ over Q is $E = Q(\sqrt[4]{3}, i)$. (You do not need to prove any of the claims in the previous sentence.)
(a) Let $\sigma \in \text{Gal}(E/Q)$ where $\sigma(\sqrt[4]{3}) = -\sqrt[4]{3}$ and $\sigma(i) = -i$. Determine with explanation the fixed field E^σ.
(b) Let $\tau \in \text{Gal}(E/Q)$ where $\tau(\sqrt[4]{3}) = i\sqrt[4]{3}$ and $\tau(i) = -i$. Determine with explanation the fixed field E^τ.

7. From Final Exam, May 2008 #4.
Let $f(x) = x^6 - 3 \in Q[x]$ and let E be the splitting field for $f(x)$ over Q.
(a) Prove $E = Q(\sqrt[6]{3}, i)$.
(b) Determine, with explanation, $|\text{Gal}(f(x)/Q)|$.
(c) Let σ be the automorphism on E which sends $\sqrt[6]{3}$ to $\omega\sqrt[6]{3}$ (where ω is a primitive 6th root of unity) and sends i to i. Determine, with explanation, the permutation on the roots of $f(x)$ determined by σ.

8. From Comp, May 2008 #5.
Let $p(x) = x^4 - 5 \in \mathbb{Q}[x]$ and let E denote the splitting field for $p(x)$ over $\mathbb{Q}$.
(a) Show that $E = \mathbb{Q}(i, \sqrt[4]{5})$.
(b) Describe all of the automorphisms of E over $\mathbb{Q}$ and explain how you know that they are all automorphisms (without directly verifying that they satisfy the definition of an automorphism).
(c) Let τ be the automorphism which sends i to i and $\sqrt[4]{5}$ to $i\sqrt[4]{5}$ and let σ be the automorphism which sends i to $-i$ and $\sqrt[4]{5}$ to $i\sqrt[4]{5}$. Determine the permutation of the roots of $p(x)$ which corresponds to each of these automorphisms. Use this information to show $\text{Gal}(E/\mathbb{Q}) \cong D_4$ (where D_4 denotes the dihedral group with 8 elements).
(d) Determine, with explanation, the fixed field of the automorphism σ (where σ is an defined in part (c)).

9. From Comp, May 2007 #3.
Let K be a splitting field for $x^8 - 4 \in \mathbb{Q}[x]$.
(a) Find $[K:\mathbb{Q}]$.
(b) Show that $\text{Gal}(K/\mathbb{Q})$ is not cyclic.

10. From Comp, May 2006 #1.
Let K be a splitting field for $x^4 + 2 \in \mathbb{Q}[x]$.
(a) Find $[K:\mathbb{Q}]$.
(b) Show that $\text{Gal}(K/\mathbb{Q})$ is not abelian.
(c) If ω is a primitive 8th root of unity, let $k \in \text{Gal}(K/\mathbb{Q})$ satisfy $k(\sqrt[4]{2}\,\omega) = \sqrt[4]{2}\,\omega^3$ and $k(\sqrt[4]{2}^{\,3}) = \sqrt[4]{2}\,\omega$. Find the fixed field of k.