

Chapter 1

Review of Factoring

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Lecture Note-Taking Guide Scoring Rubric

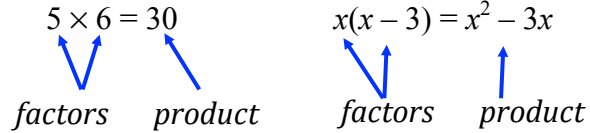
Completion of these lecture notes for extra credit points is optional. Because it is an extra credit option, the expectation of work quality is very high. There is no partial extra credit for this assignment. Any of the following criteria that are not met will result in no extra credit.

Extra Credit Points	Condition
5	<ul style="list-style-type: none">• complete• all worksteps shown• correct answers• neatly done in pencil• correctly ordered• hole punched• fastened in folders with fasteners• turned in on time• labeled with name and site
0	Any of the required criteria have not been met

1.1 Greatest Common Factor

Prime Factorization

The result of a multiplication problem is called a *product*. The numbers being multiplied are called *factors*.

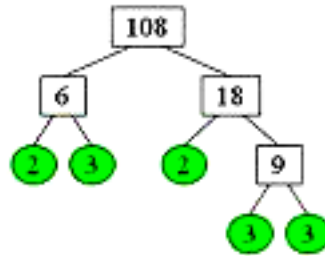


The *prime factorization* of a positive integer is the factorization in which all the factors are prime numbers. A few examples follow:

Number	Factorization	Prime Factorization
30	$6 \cdot 5$	$2 \cdot 3 \cdot 5$
4	2^2	2^2
36	$4 \cdot 9$	$2^2 \cdot 3^2$

Suppose we want to find the prime factorization of 108.

We can find prime factorizations by writing the number as a product of two factors, then writing each factor as a product of two factors, repeating this process until all the factors are prime numbers. A *factor tree* is useful in organizing this work as shown at right.



From the "leaves" of the factor tree, we have that the prime factorization of 108 is $2^2 \times 3^3$.

Or, alternatively, divide successively by prime factors as shown:

$$\begin{array}{r}
 2 \overline{)108} \\
 2 \overline{)54} \\
 3 \overline{)27} \\
 3 \overline{)9} \\
 3 \\
 \hline
 108 = 2^2 \times 3^3
 \end{array}$$

Demonstration Problems	Practice Problems
Write the prime factorization of 1. (a) 20	Write the prime factorization of 1. (b) 18
2. (a) 248	2. (b) 216
Answers: 1. (b) $2 \cdot 3^2$; 2. (b) $2^3 \cdot 3^3$	

Factoring Out the Greatest Common Factor

The Distributive Property gives us that for any real numbers a , b , and c ,

$$a(b + c) = ab + ac.$$

If we start with $ab + ac$ and write it as a product of factors, $a(b + c)$, then the process is called *factoring*.

For example, $12x^3 - 8xy = 4x \cdot 3x^2 - 4x \cdot 2y = 4x(3x^2 - 2y)$

$$\text{GCF}(12x^3, 8xy) = 4x$$

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Factor out the GCF in each expression.	Factor out the GCF in each expression.
5. (a) $2w + 4z$	5. (b) $6y + 3$
6. (a) $-2k^7m^4 + 4k^3m^6$	6. (b) $-6h^5j^2 + 3h^3j^6$
7. (a) $8a^2b^2 + 16a^2b - 24ab$	7. (b) $4a^3b^2 + 8a^2b^2 - 24ab^2$
8. (a) $a(x - 3) + b(x - 3)$	8. (b) $z(y + 4) + 3(y + 4)$
Answers: 5. (b) $3(2y + 1)$; 6. (b) $3h^3j^2(-2h^2 + j^4)$; 7. (b) $4ab^2(a^2 + 2a - 6)$ 8. (b) $(y + 4)(z + 3)$	

Box Method Factoring

Alternatively, we can factor $xy + 2y + 5x + 10$ by the "box method".

Step 1: Draw a 2×2 box.

Step 2: Place the terms in the box in the order given.

Step 3: Factor the GCF of each row and column, placing the GCF to the left of or above the box.

Step 4: Read the factors from the left side and top side of the box.



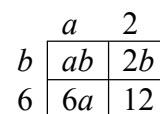
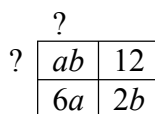
<i>Demonstration Problems</i>	<i>Practice Problems</i>
Use the box method to factor. 3. (a) $ap + 3a - p - 3$ 4. (a) $2y^2 + 3y - 16y - 24$	Use the box method to factor. 3. (b) $w^2 + aw - w - a$ 4. (b) $2x^2 - 3x - 2x + 3$
Answers: 3. (b) $(w - 1)(w + a)$; 4. (b) $(x - 1)(2x - 3)$	

Note that the terms may need to be re-ordered before the preceding methods will work. For example:

Factor $ab + 12 + 6a + 2b$.

In the order given, the GCF of the top row is 1 and the GCF of the first column is 1, but $1 \cdot 1 \neq ab$.

If we change the order so that the GCF of the top row and the first column is not 1, then the method yields the proper binomial factors.



1.3 Factoring Trinomials

Factoring the Trinomial $ax^2 + bx + c$ with $a = 1$

Notice the following 3 multiplying and simplifying patterns:

1. $(x + 4)(x + 3)$
 $x^2 + 3x + 4x + 12$
 $x^2 + 7x + 12$

2. $(x + 4)(x + 3)$
 $x^2 + 3x + 4x + 12$
 $x^2 + 7x + 12$

3. $(x + 4)(x + 3)$
 $x^2 + 3x + 4x + 12$
 $x^2 + 7x + 12$

To factor the trinomial $x^2 + 7x + 12$ into the product of two binomials, without already knowing the binomials, we need to get “clues” from the 3 terms.

From (1), we see x^2 resulted from binomials starting with $(x + \underline{\quad})(x + \underline{\quad})$.

From (2), we know that **12** is a **product** of the two constants in the binomials. But this is not a strong enough clue on its own to complete the factorization that we started from pattern (1). There are many pairs of factors whose product is 12.

$$\begin{array}{l} 1 \cdot 12 = 12 \\ 2 \cdot 6 = 12 \\ 3 \cdot 4 = 12 \end{array} \quad \text{and} \quad \begin{array}{l} -1 \cdot -12 = 12 \\ -2 \cdot -6 = 12 \\ -3 \cdot -4 = 12 \end{array}$$

From (3), we know that **7** is the **sum** of the two constants in the binomials. Since 12 is the product of the same two numbers, we can use the list of product pairs to find two numbers whose product is **12** AND whose sum is **7**.

Product is 12	Sum is 7
$1 \cdot 12 = 12$	$1 + 12 = 13$
$2 \cdot 6 = 12$	$2 + 6 = 8$
$3 \cdot 4 = 12$	$3 + 4 = 7$
$-1 \cdot -12 = 12$	$-1 + -12 = -13$
$-2 \cdot -6 = 12$	$-2 + -6 = -8$
$-3 \cdot -4 = 12$	$-3 + -4 = -7$

To save effort, a shorthand list could be created as follows:

Product 12	
1	12
2	6
3	4
Sum 7	

We see that: $3 \cdot 4 = 12$ and $3 + 4 = 7$
 So then $x^2 + 7x + 12 = (x + \underline{3})(x + \underline{4})$.

Mentally check the sum of each pair until the desired sum is found.

Consider another pattern:

$$(x + 4y)(x + 3y)$$

$$x^2 + 3xy + 4xy + 12y^2$$

$$x^2 + 7xy + 12y^2$$

From this, we can see x^2 and $12y^2$ resulted from binomials containing

$$(x + \underline{\quad}y)(x + \underline{\quad}y).$$

To complete the factoring of $x^2 + 7xy + 12y^2$, we follow the same procedure as we did in the above pages by finding two numbers whose product is 12 and sum is 7 and inserting those numbers before the y in each binomial.

<i>Demonstration Problems</i>	<i>Practice Problems</i>								
<p>Factor each trinomial.</p> <p>4. (a) $x^2 + 2xy - 8y^2$</p> <p>$= (x \underline{\quad} y)(x \underline{\quad} y)$</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin-left: auto; margin-right: auto;"> <table style="width: 100%; border-collapse: collapse;"> <tr> <td colspan="2" style="text-align: center; color: red; font-weight: bold;">Product -8</td> </tr> <tr> <td style="text-align: center; width: 50%;">-1</td> <td style="width: 50%;"></td> </tr> <tr> <td style="text-align: center;">-2</td> <td></td> </tr> <tr> <td colspan="2" style="text-align: center; color: green; font-weight: bold;">Sum 2</td> </tr> </table> </div>	Product -8		-1		-2		Sum 2		<p>Factor each trinomial.</p> <p>4. (b) $a^2 - 7ab + 10b^2$</p>
Product -8									
-1									
-2									
Sum 2									
<p style="font-size: small;">Answer: 4. (b) $(a - 2b)(a - 5b)$</p>									

To factor completely, first look for a GCF, factor it out if there is one, then factor the resulting polynomial if it can be factored.

<i>Demonstration Problems</i>	<i>Practice Problems</i>								
<p>Factor completely.</p> <p>5. (a) $6w^2 - 12w - 18$</p> <p>$= 6(w^2 - 2w - 3)$</p> <p>$= 6(w \underline{\quad})(w \underline{\quad})$</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin-left: auto; margin-right: auto;"> <table style="width: 100%; border-collapse: collapse;"> <tr> <td colspan="2" style="text-align: center; color: red; font-weight: bold;">Product -3</td> </tr> <tr> <td style="text-align: center; width: 50%;">-1</td> <td style="width: 50%;"></td> </tr> <tr> <td></td> <td></td> </tr> <tr> <td colspan="2" style="text-align: center; color: green; font-weight: bold;">Sum -2</td> </tr> </table> </div>	Product -3		-1				Sum -2		<p>Factor completely.</p> <p>5. (b) $2w^2 - 36w + 162$</p>
Product -3									
-1									
Sum -2									
<p style="font-size: small;">Answer: 5. (b) $2(w - 9)(w - 9)$</p>									

Factoring the Trinomial $ax^2 + bx + c$ with $a \neq 1$

The ac Method

Study the product at right.

$$(2x + 3)(3x + 4)$$

$$2x(3x + 4) + 3(3x + 4)$$

$$6x^2 + 8x + 9x + 12$$

$$6x^2 + 17x + 12$$

Notice $6 \cdot 12 = 72 = 9 \cdot 8$

To factor the trinomial $6x^2 + 17x + 12$ into the product of two binomials, without already knowing the binomials, we need to get “clues” from the 3 terms.

If we find two numbers whose product is **72** and whose sum is **17**, we’ll be able to rewrite the trinomial as the polynomial with 4 terms the we see above and then factor by grouping or box method.

Product 72	
1	72
2	36
3	24
4	18
6	12
8	9
Sum 17	

This gives us that

$$\begin{aligned} 6x^2 + 17x + 12 &= 6x^2 + 8x + 9x + 12 \\ &= 2x(3x + 4) + 3(3x + 4) \\ &= (2x + 3)(3x + 4) \end{aligned}$$

or

	3x	4
2x	$6x^2$	$8x$
3	$9x$	12

In general, to factor the trinomial $ax^2 + bx + c$,

Step 1: Find two numbers that have a product of ac and a sum of b .

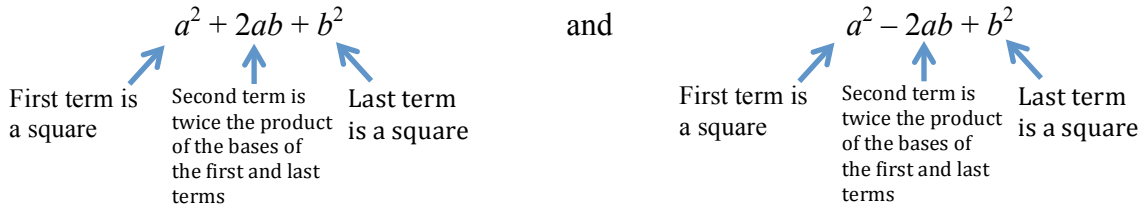
Step 2: Replace bx by the sum of the two terms with **coefficients** that are the two numbers found in Step 1.

Step 3: Factor the resulting four-term polynomial by grouping or by the box method.

Special Trinomials

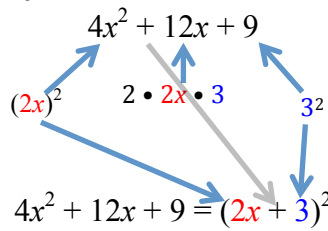
Since $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$, and
 $(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$,

we consider trinomials of the following form to be to be *perfect square trinomials*:



If we can verify that a trinomial is a perfect square at a glance, then we can skip the lengthy techniques for factoring trinomials and write the factors immediately.

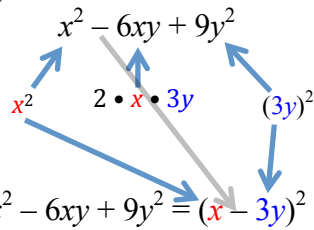
Example (a) Factor $4x^2 + 12x + 9$



Thus,

$$4x^2 + 12x + 9 = (2x + 3)^2$$

Example (b) Factor $x^2 - 6xy + 9y^2$



Thus,

$$x^2 - 6xy + 9y^2 = (x - 3y)^2$$

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Factor the following trinomials. 9. (a) $16a^2 + 8a + 1$ 10. (a) $36y^2 - 12yz + z^2$	Factor the following trinomials. 9. (b) $25x^2 + 30x + 9$ 10. (b) $9a^2 - 6ab + b^2$
Answers: 9. (b) $(5x + 3)^2$; 10. (b) $(3a - b)^2$	

1.4 Factoring Binomials

Difference of Two Squares

Complete the table and notice the pattern that emerges.

	F	+	O	+	I	+	L	
$(x + 1)(x - 1) =$	x^2	+	$-1 \cdot x$	+	$1 \cdot x$	+	$-1 \cdot 1$	$= x^2 - 1$
$(2a - 3)(2a + 3) =$		+		+		+		
$(4m + 5n)(4m - 5n) =$		+		+		+		

Recall that the result of a subtraction problem is called a *difference*, so then $a^2 - b^2$ is a *difference of squares*.

$$a^2 - b^2 = (a + b)(a - b)$$

When we verify that a binomial is a difference of squares, we can use this pattern to write the factors immediately.

Example (a) Factor $x^2 - 9$.

$$\begin{aligned} x^2 - 9 \\ = x^2 - 3^2 \\ = (x + 3)(x - 3) \end{aligned}$$

Example (b) Factor $4x^2 - 25$.

$$\begin{aligned} 4x^2 - 25 \\ = (2x)^2 - 5^2 \\ = (2x + 5)(2x - 5) \end{aligned}$$

Example (c) Factor $x^4 - 9y^2$.

$$\begin{aligned} x^4 - 9y^2 \\ = (x^2)^2 - (3y)^2 \\ = (x^2 + 3y)(x^2 - 3y) \end{aligned}$$

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Factor each polynomial. 1. (a) $a^2 - 4$	Factor each polynomial. 1. (b) $x^2 - 49$
2. (a) $y^2 - 9x^2$	2. (b) $16x^2 - y^2$
3. (a) $144n^2 - 1$	3. (b) $1 - 25w^2$
4. (a) $9a^2 - 16b^2$	4. (b) $25a^2 - 81b^2$
5. (a) $9a^4 - 16b^2$	5. (b) $25a^4 - 81b^2$
Answers: 1. (b) $(x + 7)(x - 7)$; 2. (b) $(4x + y)(4x - y)$; 3. (b) $(1 + 5w)(1 - 5w)$; 4. (b) $(5a + 9b)(5a - 9b)$; 5. (b) $(5a^2 + 9b)(5a^2 - 9b)$;	

Sum of Cubes and Difference of Cubes

Multiply the following and notice the pattern that emerges.

$(x + 1)(x^2 - x + 1) =$	$x \cdot x^2 + 1 \cdot x^2 + x \cdot -x + 1 \cdot -x + x \cdot 1 + 1 \cdot 1$	$= x^3 + 1$
$(3a + 5)(9a^2 - 15a + 25) =$	$+ \quad + \quad + \quad + \quad +$	
$(x - 1)(x^2 + x + 1) =$	$x \cdot x^2 + -1 \cdot x^2 + x \cdot x + -1 \cdot x + x \cdot 1 + -1 \cdot 1$	$= x^3 - 1$
$(a - 4)(a^2 + 4a + 16) =$	$+ \quad + \quad + \quad + \quad +$	

Factoring Patterns	
<i>Sum of Cubes</i>	<i>Difference of Cubes</i>
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

When we verify the binomial is a sum or a difference of cubes can use these patterns to write the factors immediately.

Example (d) Factor $x^3 + 27$.

$$\begin{aligned}
 & x^3 + 27 \\
 & = x^3 + 3^3 \\
 & = (x + 3)(x^2 - 3x + 3^2) \\
 & = (x + 3)(x^2 - 3x + 9)
 \end{aligned}$$

Example (e) Factor $8x^3 - 125$.

$$\begin{aligned}
 & 8x^3 - 125 \\
 & = (2x)^3 - 5^3 \\
 & = (2x - 5)((2x)^2 + 2x \cdot 5 + 5^2) \\
 & = (2x - 5)(4x^2 + 10x + 25)
 \end{aligned}$$

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Factor each binomial. 6. (a) $a^3 + 64$	Factor each binomial. 6. (b) $x^3 + 8$
7. (a) $y^3 - 27x^3$	7. (b) $8x^3 - y^3$
8. (a) $27n^3 - 1$	8. (b) $1 - 125w^3$
Answers: 6. (b) $(x + 2)(x^2 - 2x + 4)$; 7. (b) $(2x - y)(4x^2 + 2xy + y^2)$; 8. (b) $(1 - 5w)(1 + 5w + 25w^2)$	

Factoring Completely**Example (f)** Factor $2x^3 - 54$.

$$\begin{aligned}
 &2x^3 - 54 \\
 &= 2(x^3 - 27) \\
 &= 2(x - 3)(x^2 + 3x + 9)
 \end{aligned}$$

Example (g) Factor $x^6 - y^6$.

$$\begin{aligned}
 &x^6 - y^6 \\
 &= (x^3)^2 - (y^3)^2 \\
 &= (x^3 + y^3)(x^3 - y^3) \\
 &= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)
 \end{aligned}$$

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Factor each polynomial. 9. (a) $a^4 - 16$ 10. (a) $3y^6 - 192x^6$	Factor each polynomial. 9. (b) $x^4 - 81$ 10. (b) $2x^6 - 128y^6$
Answers: 9. (b) $(x^2 + 9)(x + 3)(x - 3)$; 10. (b) $2(x - 2y)(x^2 + 2xy + 4y^2)(x + 2y)(x^2 - 2xy + 4y^2)$	

1.5 Choosing a Factoring Method

Factoring Methods for Polynomials of 2 – 4 Terms									
<i>Number of terms</i>	<i>Step 1</i>	<i>Identify the polynomial or polynomial factor</i>	<i>Step 2</i>						
2	Factor out any GCF	Difference of squares \Rightarrow	$a^2 - b^2 = (a + b)(a - b)$						
		Sum of cubes \Rightarrow	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$						
		Difference of cubes \Rightarrow	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$						
		None of the above \Rightarrow	Cannot be factored by methods learned in this chapter.						
3	Factor out any GCF	$x^2 + bx + c \Rightarrow$	$= (x \quad)(x \quad)$ <table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td colspan="2">Product c</td></tr> <tr><td> </td><td> </td></tr> <tr><td colspan="2">Sum b</td></tr> </table>	Product c				Sum b	
		Product c							
Sum b									
$ax^2 + bx + c \Rightarrow$	$ax^2 + bx + c$ $= ax^2 + b_1x + b_2x + c$ <table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td colspan="2">Product ac</td></tr> <tr><td>b_1</td><td>b_2</td></tr> <tr><td colspan="2">Sum b</td></tr> </table> <p>Then factor by grouping or box method.</p>	Product ac		b_1	b_2	Sum b			
Product ac									
b_1	b_2								
Sum b									
None of the above \Rightarrow	Cannot be factored by methods learned in this chapter.								
4	Factor out any GCF		Try to factor by grouping or box method.						

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Factor each polynomial. 1. (a) $a^2 - 25$ 2. (a) $8x^2 + 26x + 15$ 3. (a) $n^3 - 1$ 4. (a) $2a^4 - 32b^4$ 5. (a) $4ab + a - 1 - 4b$	Factor each polynomial. 1. (b) $x^2 - 16$ 2. (b) $6x^2 + x - 2$ 3. (b) $1 - w^3$ 4. (b) $243x^4 - 3y^4$ 5. (b) $2x^3 + x^2 - 50x - 25$
Answers: 1. (b) $(x + 4)(x - 4)$; 2. (b) $(3x + 2)(2x - 1)$; 3. (b) $(1 - w)(1 + w + w^2)$; 4. (b) $3(9x^2 + y^2)(3x + y)(3x - y)$; 5. (b) $(x + 5)(x - 5)(2x + 1)$	

1.6 Solving Equations by Factoring

Solve the following equations for x .

$2x = 0$	$-5x = 0$
$\frac{2}{3}x = 0$	$ax = 0$

If any two numbers have a product of zero, then one or both of them must be what number?

The answer to the preceding question can be formalized as

Zero Product Property*
If $ab = 0$, then $a = 0$ or $b = 0$.

* This property can be extended to any finite number of factors.

And now consider the equation

$$(x - 1)(x + 2) = 0$$

Can you find a solution by inspection (guessing and checking)?

Can you find a second solution?

Since $(x - 1)(x + 2) = 0$ is a product equal to 0, we can apply the Zero Product Property to solve it.

By the Zero Product Property, if $(x - 1)(x + 2) = 0$, then

$$\begin{array}{l} x - 1 = 0 \quad \text{or} \quad x + 2 = 0 \\ x = 1 \quad \text{or} \quad x = -2 \end{array}$$

Solution set: $\{1, -2\}$

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Solve 1. (a) $(x - 3)(x - 5) = 0$ 2. (a) $(2x - 1)(3x - 4) = 0$ 3. (a) $x(x + 5) = 0$	Solve 1. (b) $(x + 1)(x - 2) = 0$ 2. (b) $(3x - 5)(4x - 1) = 0$ 3. (b) $x(x - 7) = 0$
Answers: 1. (b) $\{-1, 2\}$; 2. (b) $\{\frac{5}{3}, \frac{1}{4}\}$; 3. (b) $\{0, 7\}$	

<i>Demonstration Problems</i>	<i>Practice Problems</i>
Solve 7. (a) $x^2 = 6x$	Solve 7. (b) $x^2 = -5x$
8. (a) $x^2 + x = 56$	8. (b) $x^2 + 2x = 8$
9. (a) $5x^2 - 20 = 0$	9. (b) $3x^2 - 3 = 0$
Answers: 7. (b) $\{0, -5\}$; 8. (b) $\{-4, 2\}$; 9. (b) $\{1, -1\}$;	

