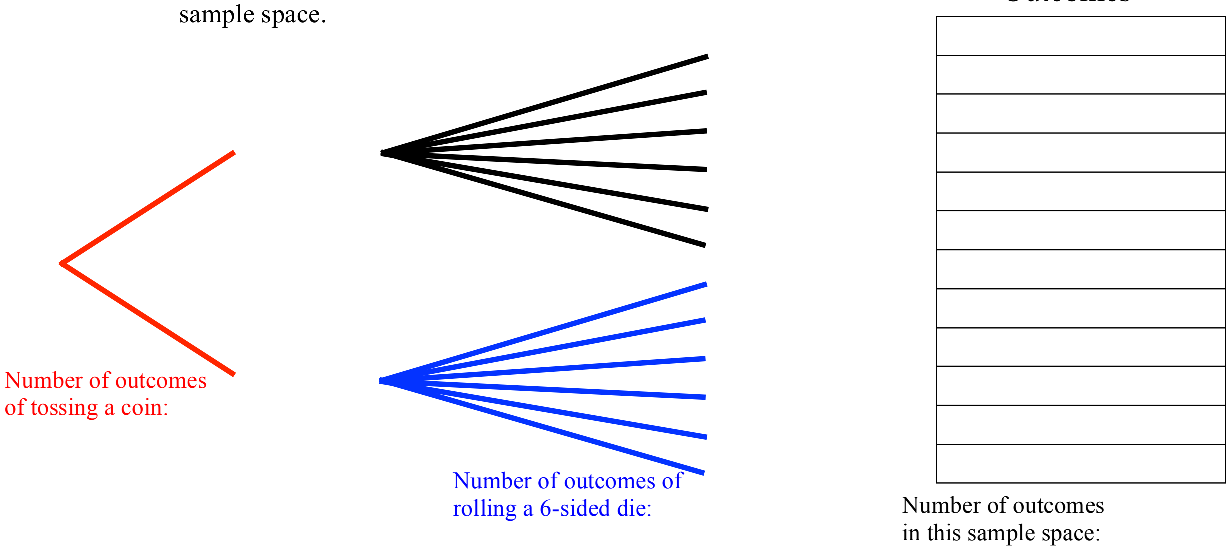
3.6 Counting Rules

Example 1: Suppose a lottery game designer wants to list all possible outcomes of the following sequences of events:

a. tossing a coin once and rolling a 6-sided die once

Complete the tree diagram branch labels and list the outcomes in the sample space.

Outcomes



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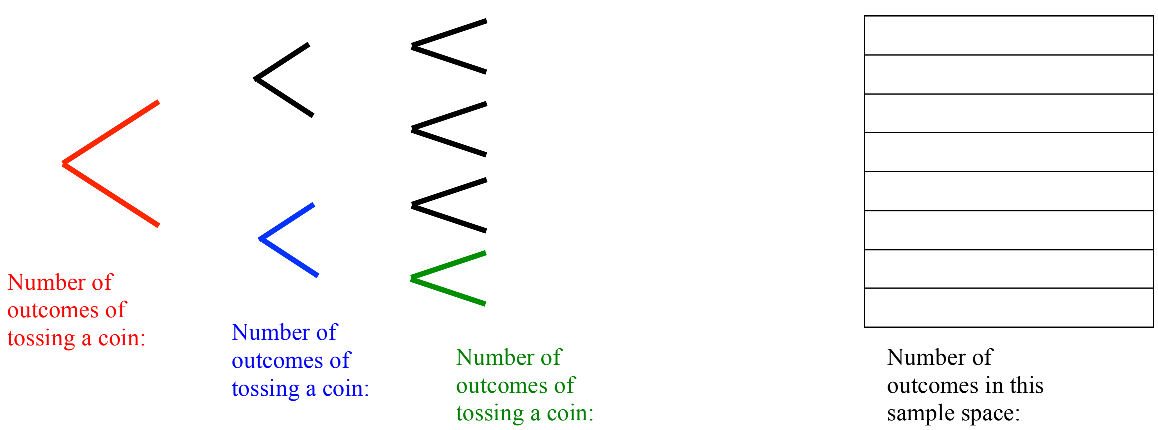
b. tossing a coin 3 times.

Outcomes



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To find certain probabilities, we don't need a list of all outcomes in the event and sample spaces, but we do need to know the number of outcomes in each. The examples on the previous page illustrate the following principle.

Fundamental Counting Principle
In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 possibilities, and so forth, the total number of possibilities of the sequence will be $k_1 \cdot k_2 \cdot k_3 \cdot \cdots \cdot k_n$

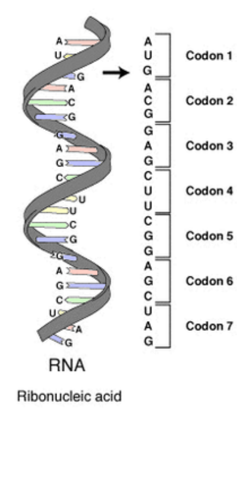
Example 2: Use the fundamental counting principle to determine the number of student classifications according to gender (male, female), academic level (high school, freshman, sophomore), and enrollment in statistics (enrolled, not enrolled).

Gender	**Academic level**	**Enrollment in statistics**	
• male • female	• high school • freshman • sophomore	• enrolled • not enrolled	Product
•	•	=	

Example 3: Use the fundamental counting principle to determine the number of license plates that can be made using one digit followed by 3 letters, followed by 3 digits

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	
0 1 2 3 4 5 6 7 8 9	A N B O C P D Q E R F S G T H U I V J W K X L Y M Z	A N B O C P D Q E R F S G T H U I V J W K X L Y M Z	A N B O C P D Q E R F S G T H U I V J W K X L Y M Z	0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9	Product
•	•	•	•	•	•	=	

Example 4: Use the fundamental counting principle to determine the number of distinct codons that can be formed from the nucleotides, A, U, G, and C, taken 3 at a time.



Example 5: Use the fundamental counting principle to determine the number of cell phone customers that can be served if the area code and first three digits must be 530 727 or 530 787

(530) 727 – xxxx				or	(530) 787 - xxxx				
1st	2nd	3rd	4th		1st	2nd	3rd	4th	
0	0	0	0		0	0	0	0	
1	1	1	1		1	1	1	1	
2	2	2	2		2	2	2	2	
3	3	3	3		3	3	3	3	
4	4	4	4		4	4	4	4	
5	5	5	5		5	5	5	5	
6	6	6	6		6	6	6	6	
7	7	7	7		7	7	7	7	
8	8	8	8		8	8	8	8	
9	9	9	9		9	9	9	9	Total
	•	•	•	+		•	•	•	=

Example 6: the number of 5-digit ID numbers that can be made if no digit can be repeated

1st	2nd	3rd	4th	5th	
0	0	0	0	0	
1	1	1	1	1	
2	2	2	2	2	
3	3	3	3	3	
4	4	4	4	4	
5	5	5	5	5	
6	6	6	6	6	
7	7	7	7	7	
8	8	8	8	8	
9	9	9	9	9	
	(take 1 out)	(take 2 out)	(take 3 out)	(take 4 out)	Product
	•	•	•	•	=

Example 7: the number of 10-digit ID numbers if no digit can be repeated

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	
	0	0	0	0	0	0	0	0	0	
0	1	1	1	1	1	1	1	1	1	
1	2	2	2	2	2	2	2	2	2	
2	3	3	3	3	3	3	3	3	3	
3	4	4	4	4	4	4	4	4	4	
4	5	5	5	5	5	5	5	5	5	
5	6	6	6	6	6	6	6	6	6	
6	7	7	7	7	7	7	7	7	7	
7	8	8	8	8	8	8	8	8	8	
8	9	9	9	9	9	9	9	9	9	
9	(take 1 out)	(take 2 out)	(take 3 out)	(take 4 out)	(take 5 out)	(take 6 out)	(take 7 out)	(take 8 out)	(take 9 out)	Product
	•	•	•	•	•	•	•	•	•	=

Suppose we had a set of 100 distinct symbols and wanted to find the number of ID numbers that could be made if no repeats are allowed. We would need to calculate
$100 \bullet 99 \bullet 98 \bullet 97 \bullet \cdots \bullet 3 \bullet 2 \bullet 1$. It is cumbersome to write this in its entirety. Thus, mathematicians have created a shorthand called factorial notation.

We write

$$100! = 100 \bullet 99 \bullet 98 \bullet 97 \bullet \cdots \bullet 3 \bullet 2 \bullet 1$$

(100! is read as 100 factorial)

Factorial notation will be used extensively in our work in permutations and combinations.

Note: We define $0! = 1$.

The previous problem of finding the number of 10-digit ID numbers for which no repeats are allowed is an example of a permutation. It is the number of possible arrangements of 10 digits in a specific order.

In general, a *permutation* is an arrangement of n objects in a specific order.

Example 8: Suppose a rancher has five horse and five stalls. How many ways can the rancher place the horses into the stalls?

Let's call the horses A, B, C, D, and E.

1^{st} stall	2^{nd} stall	3^{rd} stall	4^{th} stall	5^{th} stall	
A B C D E	A B C D E (take 1 out)	A B C D E (take 2 out)	A B C D E (take 3 out)	A B C D E (take 4 out)	Product
•	•	•	•	=	

There are $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 120$ ways to place the horses.

Example 9: Suppose there are only 3 stalls. How many ways can we choose from the 5 horses to place them into 3 stalls?

1^{st} stall	2^{nd} stall	3^{rd} stall	
A B C D E	A B C D E (take 1 out)	A B C D E (take 2 out)	Product
•	•	=	

Example 10: How many ways can we choose 3 horses from 10 to place them into 3 stalls? (We will continue to name them using consecutive letters.)

1^{st} stall	2^{nd} stall	3^{rd} stall	
A B C D E F G H I J	A B C D E F G H I J (take 1 out)	A B C D E F G H I J (take 2 out)	Product
•	•	=	

Example 11: How many ways can we choose 6 horses from 7 to place them in 6 stalls?

1^{st} stall	2^{nd} stall	3^{rd} stall	4^{th} stall	5^{th} stall	6^{th} stall	
A B C D E F G	A B C D E F G (take 1 out)	A B C D E F G (take 2 out)	A B C D E F G (take 3 out)	A B C D E F G (take 4 out)	A B C D E F G (take 5 out)	Product
•	•	•	•	•	=	

Notice that $5 \bullet 4 \bullet 3 = \dfrac{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{2 \bullet 1} = \dfrac{5!}{2!} = \dfrac{5!}{(5-3)!}$ and

$10 \bullet 9 \bullet 8 = \dfrac{10 \bullet 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1} = \dfrac{10!}{7!} = \dfrac{10!}{(10-3)!}$ and

$7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 = \dfrac{7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{1} = \dfrac{7!}{1!} = \dfrac{7!}{(7-6)!}$.

We can use this emerging pattern to write a formula for finding arrangements of n objects taken r at a time.

Permutation Rule
The arrangement of n objects in a specific order using r objects at a time is called a *permutation of n objects taken r objects at a time.* It is written as ${}_nP_r$, and the formula is $${}_nP_r = \frac{n!}{(n-r)!}$$

Example 12: Use the permutation rule to determine the number of distinct permutations possible of a padlock in which 3 numbers from 0 – 39, no repetitions, that can be used to form the code to unlock the padlock.

Example 13: Let's revisit our 5 horses. Suppose we want to select 2 of them to turn out into a pasture. How many ways can we select 2 of the 5 horses for this purpose?

If we were placing them in 2 stalls, we would use the formula

$$_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{3 \bullet 2 \bullet 1} = 5 \bullet 4 = 20 .$$

However, the formula assumes that the order in which the selection is made matters. Let's list the 20 permutations.

AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
BA	CA	DA	EA	CB	DB	EB	DC	EC	ED

In choosing 2 of the 5 horses to turn out into the pasture, we count

AB the same as BA,
AC the same as CA,
AD the same as DA,

and so on. Thus, there are 20 ÷ 2 = 10 ways to choose from 5 horses, 2 at a time without regard to order. We call this kind of selection a combination.

A *combination* is a selection of objects without regard to order.

Let's develop a formula for finding combinations of n objects selected r at a time.

Example 14: How many ways can we choose 5 horses 3 at a time without regard to order?

There are $_5P_3 = 60$ permutations of choosing 5 horses 3 at a time if order matters. Since order does not matter, we divide out the duplicates. But how many duplicates do we have when choosing 3 at a time?

Here is a list of the 60 permutations.

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	CAD	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	CDA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

Notice that the arrangements in each column all represent the permutations of 3 objects taken 3 at a time. Each column contains $_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3!}{1} = 3!$

To find the number of ways to choose 5 horses 3 at a time without regard to order, we should divide 60 by what number?

Example 15: How many ways can we choose 8 horses 5 at a time without regard to order?

There are ${}_8P_5 = 6720$ permutations of choosing 8 horses 5 at a time if order matters. Order does not matter, so we divide out the duplicates, ${}_5P_5 = r!$.

Let's develop a formula now.

We will be using the notation ${}_nC_r$ to represent the number of combinations of choosing n objects r at a time. We have found that

$${}_nC_r = {}_nP_r \div r! = \frac{n!}{(n-r)!} \div r! = \frac{n!}{(n-r)!r!}$$

Combination Rule
The number of selections possible of r objects chosen from n objects is called a *combination of n objects taken r objects at a time.* It is written as ${}_nC_r$, and the formula is $${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example 16: Use the combination rule to determine how many ways 3 candies can be selected from a dish of 8 candies if order is to be disregarded.

Example 17: Use the combination rule to determine how many ways can a jury of 6 women and 6 men be selected from a pool of 11 women and 9 men?

Combinations and Permutations

What's the Difference?

In English we use the word "combination" loosely, without thinking if the **order** of things is important. In other words:

"My fruit salad is a combination of apples, grapes and bananas" We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

"The combination to the safe was 472". Now we **do** care about the order. "724" would not work, nor would "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more *precise* language:

- If the order doesn't matter, it is a **Combination**.
- If the order **does** matter it is a **Permutation**.

In other words:

A Permutation is an **ordered** Combination.

Label each as Permutation (P) or Combination (C) problems.

a.	How many ways can we choose 20 2×4's from a stack of 30 2×4's?	
b.	How many ways can 3 books be arranged on a shelf?	
c.	How many ways can 6 of 8 dogs be placed in 6 kennel cages?	
d.	How many ways can 6 of 8 dogs be turned out into a play yard?	
e.	How many ways can 3 of 5 flowers be given to 3 people?	
f.	How many ways can 2 candidates be chosen from 5 candidates for final interviews?	
g.	How many license plates can be made using 3 letters followed by 3 numbers?	
h.	How many ways can 5 radio commercials be run during an hour	

Demonstration Problems	*Practice Problems*
Evaluate each expression. **1. (a)** $8!$	Evaluate each expression. **1. (b)** $7!$
2. (a) $\frac{8!}{5!}$	**2. (b)** $\frac{7!}{3!}$
3. (a) ${}_8P_3$	**3. (b)** ${}_7P_4$
4. (a) ${}_3P_3$	**4. (b)** ${}_4P_4$
5. (a) ${}_8C_3$	**5. (b)** ${}_7C_4$

Answers: **1. (b)** 5040; **2. (b)** 840; **3. (b)** 840; **4. (b)** 5040; **5. (b)** 35

Demonstration Problems	*Practice Problems*
6. (a) How many different ID card numbers can be made if there are 6 digits?	**6. (b)** How many different ID card numbers can be made if there are 5 digits?
7. (a) How many license plates can be made if the first 3 characters are letters and the next 4 are digits?	**7. (b)** How many license plates can be made if the first 2 characters are letters and the next 5 are digits?
8. (a) How many different ID card numbers can be made if there are 6 digits and no digit can be used more than once?	**8. (b)** How many different ID card numbers can be made if there are 5 digits and no digits can be used more than once?
9. (a) How many ways can 7 dogs be chosen from 10 dogs to be placed into 7 kennels?	**9. (b)** How many ways can 5 dogs be chosen from 12 dogs to be placed into 5 kennels?
10. (a) How many ways can 10 dogs in a shelter be chosen to be turned out into the play yard 2 at a time?	**10. (b)** How many ways can 12 dogs in a shelter be chosen to be turned out into the play yard 3 at a time?
Answers: **6. (b)** 100,000; **7. (b)** 67,600,000; **8. (b)** 30,240; **9. (b)** 95,040; **10. (b)** 220	

Visit https://en.wikipedia.org/wiki/Vehicle_registration_plates_of_California to learn more about the history and future of CA license plates.