4.2 Mean, Expected Value, and Standard Deviation

Mean

Recall the formula from section 3.2 for find the population mean of a data set of N elements	$\mu = \frac{x_1 + x_2 + x_3 + \cdots + x_N}{N} = \frac{\sum_{i=1}^{N} x_i}{N}$

We can find the mean of the data set of a probability distribution using this formula when the data set is small.

Example 1: Let's look at the sample space of the probability experiment of tossing a coin three times. Let X = the number of heads.

TTT ($x_1 = 0$)	HTT ($x_5 = 1$)
TTH ($x_2 = 1$)	HTH ($x_6 = 2$)
THT ($x_3 = 1$)	HHT ($x_7 = 2$)
THH ($x_4 = 2$)	HHH ($x_8 = 3$)

$$\mu = \frac{\quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad}{\quad} = \frac{\quad}{\quad} =$$

This means that when we toss a coin 3 times, we expect to see about 1.5 heads.

Can the outcome of an experiment ever be exactly 1.5 heads?

Of course not. This is a theoretical average. Since heads should occur in about one half of the tosses, we accept 1.5 as a meaningful average for our three tosses.

For most probability experiments this formula is either tedious or impossible to use. For some experiments, the sample space has an infinite number of data elements and we cannot divide a number by infinity. But we can use the following formula:

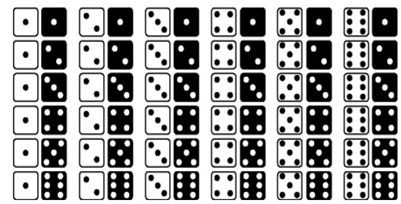
Formula for the Mean of a Probability Distribution	
The mean of a probability distribution where the random variable, X, is discrete is	$\mu = X_1 \bullet P(X_1) + X_2 \bullet P(X_2) + X_3 \bullet P(X_3) + \cdots + X_N \bullet P(X_N)$ $= \Sigma\left(X \bullet P(X)\right)$

Let's rework Example 1 using this new formula for the mean of a probability distribution:

No. of Heads X	Probability $P(X)$	$X \bullet P(X)$
$X_1 = 0$	$P(X_1) = \frac{1}{8}$	
$X_2 = 1$	$P(X_2) = \frac{3}{8}$	
$X_3 = 2$	$P(X_3) = \frac{3}{8}$	
$X_4 = 3$	$P(X_4) = \frac{1}{8}$	
	$\mu =$	

Example 2: Find the mean number of dots, X, that appear when two dice are tossed.

X	$P(X)$	$X \bullet P(X)$
2	$\frac{1}{36}$	$2 \bullet \frac{1}{36} = \frac{2}{36}$
3	$\frac{2}{36}$	
4		
$\mu =$		



Example 3: In a randomly chosen family with three children, find the mean of the number of children that are expected to be girls.

No. of Girls X	Probability $P(X)$	$X \bullet P(X)$
0		
1		
2		
3		
$\mu =$		

GGG	GGB
GBG	GBB
BGG	BGB
BBG	BBB

Example 4: In the cafeteria, the following probability distribution was obtained for the number of extras (avocado, cheese, deli meat, bacon, gluten-free bread) a person ordered with a deli sandwich. Find the mean number of extras.

No. of Extras X	Probability $P(X)$	$X \bullet P(X)$
0	.2	
1	.35	
2	.25	
3	.1	
4	.05	
5	.05	
	$\mu =$	

Variance and Standard Deviation

Recall the formulas from section 3.3 for finding the population variance and standard deviation of a data set of N elements

$$\sigma^2 = \frac{(X_1-\mu)^2+(X_2-\mu)^2+\cdots+(X_N-\mu)^2}{N} = \frac{\Sigma(X-\mu)^2}{N}$$

$$\sigma = \sqrt{\frac{\Sigma(X-\mu)^2}{N}}$$

Example 5: (Example 1 revisited): Let's find the variance and standard deviation of tossing a coin three times where X = the number of heads.

TTT ($x_1 = 0$)	HTT ($x_5 = 1$)
TTH ($x_2 = 1$)	HTH ($x_6 = 2$)
THT ($x_3 = 1$)	HHT ($x_7 = 2$)
THH ($x_4 = 2$)	HHH ($x_8 = 3$)

$$\sigma^2 = \frac{(\quad)^2+(\quad)^2+(\quad)^2+(\quad)^2+(\quad)^2+(\quad)^2+(\quad)^2+(\quad)^2}{(\quad)}$$

$\sigma =$

Again, we have a more convenient formula for finding the variance of a probability distribution that will allow us to compute the variance for large data sets and infinite data sets.

Formula for the Variance of a Probability Distribution	
The variance of a probability distribution where the random variable, X, is discrete is	$\sigma^2 = (X_1-\mu)^2 \bullet P(X_1) + (X_2-\mu)^2 \bullet P(X_2) + \cdots + (X_N-\mu)^2 \bullet P(X_N)$ $= \Sigma\left[(X-\mu)^2 \bullet P(X)\right]$

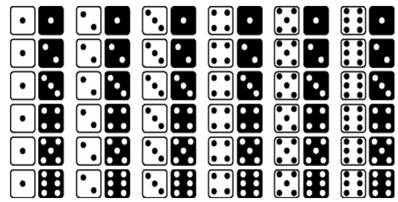
Let's rework Example 1 using this new formula for the variance of a probability distribution:

No. of Heads X	Probability $P(X)$	$X \bullet P(X)$	$(X-\mu)^2 \bullet P(X)$
0	$\frac{1}{8}$		
1	$\frac{3}{8}$		
2	$\frac{3}{8}$		
3	$\frac{1}{8}$		
		$\mu =$	$\sigma^2 =$
			$\sigma =$

Example 6: (Example 2 revisited)

Find the variance and standard deviation of the number of dots, X, that appear when two dice are tossed.

X	$P(X)$	$X \bullet P(X)$	$(X - \mu)^2 \bullet P(X)$
2	$\frac{1}{36}$	$\frac{2}{36}$	
3	$\frac{2}{36}$	$\frac{6}{36}$	
4	$\frac{3}{36}$	$\frac{12}{36}$	
5	$\frac{4}{36}$	$\frac{20}{36}$	
6	$\frac{5}{36}$	$\frac{30}{36}$	
7	$\frac{6}{36}$	$\frac{42}{36}$	
8	$\frac{5}{36}$	$\frac{40}{36}$	
9	$\frac{4}{36}$	$\frac{36}{36}$	
10	$\frac{3}{36}$	$\frac{30}{36}$	
11	$\frac{2}{36}$	$\frac{22}{36}$	
12	$\frac{1}{36}$	$\frac{12}{36}$	
		$\mu =$ 7	$\sigma^2 =$
			$\sigma =$



Expected Value

Formula for the Expected Value of a Probability Distribution	
The expected value of a probability distribution where the random variable, *X*, is discrete is	$E(X) = X_1 \bullet P(X_1) + X_2 \bullet P(X_2) + X_3 \bullet P(X_3) + \cdots + X_N \bullet P(X_N)$ $= \Sigma\left(X \bullet P(X)\right)$

Example 7: One thousand tickets are sold at $1 each for a color television valued at $350. What is the expected value of the gain if a person purchases one ticket?

Outcome	Gain X	Probability $P(X)$	$X \bullet P(X)$
Wins	$x_1 =$ $349	$P(x_1) = \frac{1}{1000}$	
Does not win	$x_2 =$ –$1	$P(x_2) = \frac{999}{1000}$	
			$E(X) =$

Example 8: A healthy 55 year old person will pay $7000 for a $500,000 life insurance policy over a period of 10 years and has a probability of living until age 65 of 0.995. Find the company's expected value of the policy.

Outcome	Company Gain X	Probability $P(X)$	$X \bullet P(X)$
Person lives	$7000		
Person dies	–$493,000		
			$E(X) =$

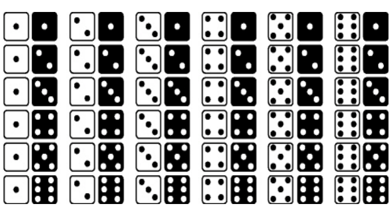
Example 9: Suppose a game of tossing a coin 3 times costs $1 to play and pays $1 for 1 head, $4 for 2 heads, and $5 for 3 heads. What is the expected net winning of playing the game once?

TTT ($x_1 = 0$)	HTT ($x_5 = 1$)
TTH ($x_2 = 1$)	HTH ($x_6 = 2$)
THT ($x_3 = 1$)	HHT ($x_7 = 2$)
THH ($x_4 = 2$)	HHH ($x_8 = 3$)

Number of Heads	Payoff	Net Payoff X	$P(X)$	$X \cdot P(X)$
0	$0	–$1	1/8	–$0.13
1	$1			
2	$4			
3	$5			
			E(net winnings) =	

In a fair game, the expected value of the net winnings will be $0. Is this a fair game?

Example 11: In a dice game, if a player rolls a sum of 12, the player wins $10. What should be the cost to play if the game is to be fair?



Sum of 12?	Payoff	Net Payoff X	$P(X)$	$X \cdot P(X)$
Yes	$10			
No	$0			
			E(net winnings) =	