210A, Final Exam, Fall 2009

Group Theory

1. Let G be a group such that $|G| = 275 = 5^2 \bullet 11$.

(a) Prove that G is not simple.
(b) Prove that if G has a normal subgroup of order 25, then G is Abelian.
(c) Determine, up to isomorphism, all Abelian groups of order 275.
(d) If the normal subgroup of order 25 in G is not cyclic, determine the isomorphism class of G.

Please complete **three** of the following four problems.

2. (a) Let $H \leq S_n$ such that $|H| = 2$. Prove that H is not normal in S_n for $n \geq 3$.
(b) Prove there does not exist a homomorphism from S_n onto A_n.

3. (a) State Cauchy's Theorem
(b) Assume Cauchy's Theorem holds for Abelian groups. Prove Cauchy's Theorem for non-abelian groups.

4. Let G be a group and X a subset of G. Recall that $\langle X \rangle$ is the subgroup generated by X. By definition, it is the smallest subgroup containing X and hence consists of all possible products of elements in X.
(a) Let $S = \{xyx^{-1}y^{-1} | \, x, y \in G\}$ and let $N = \langle S \rangle$. Prove N is a normal subgroup of G.
(b) Prove G/N is Abelian.

5. (a) Let G be a group of order 60. If G has a non-normal subgroup of order 5, prove that G is isomorphic to a subgroup of S_{12}.
(b) Prove or disprove: A simple group of order 60 is isomorphic to a subgroup of S_{12}.

Ring Theory – Please complete **three** of the following four problems.

6. Let R be a commutative ring with unity.
(a) Prove that if $a = bu$ for some $u \in U(R)$, then $(a) = (b)$.
(b) Prove that if R is an integral domain and $(a) = (b)$, then there exists $u \in U(R)$ such that $a = bu$.
(c) Prove that R is a field if and only if for every non-zero $a \in R$, $(a) = R$.

7. Let I, J be ideals of a ring R.
(a) Define $I + J = \{a + b | a \in I, b \in J\}$. Prove $I + J$ is an ideal of R.
(b) Prove that if $I + J = R$ and $\mathrm{I} \cap \mathrm{J} = \{0\}$, then $R/I \cong J$.

8. Let F be a field.
(a) Prove that for any ideal J in $F[x]$, there exists $h(x) \in F[x]$ such that $J = (h(x))$.
(b) Let $I = \{g(x) \in F[x] | f(1) = 0\}$. Prove I is an ideal of $F[x]$.
(c) Find (with justification) $h(x)$ such that $I = (h(x))$.

9. Let R be an integral domain.
(a) Without using the fact that R has a unity, prove that Frac(R) has a unity.
(b) Let K be a field. Prove that if $R \subseteq K$, then K contains an isomorphic copy of Frac(R).