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Math 210A
HW #1, 2.2 p. 50 #2, 7, 9, 11, 15

2.2 If $\alpha \in S_n$, prove that sgn (α^{-1}) = sgn(α).

Proof:

If $\alpha = \beta_1 \beta_2 \cdots \beta_k$ (a complete factorization into disjoint cycles), then
$\alpha^{-1} = \beta_k^{-1} \cdots \beta_2^{-1} \beta_1^{-1}$ (Proposition 2.4).
By definition, sgn $(\alpha^{-1}) = (-1)^{n-k}$ = sgn(α).

2.7 Given $X = \{1, 2, \ldots, n\}$, let us call a permutation τ of X an **adjacency** if it is a transposition of the form $(i, i+1)$ for $i < n$.
(i) Prove that every permutation in S_n, for $n \geq 2$, is a product of adjacencies.

Proof:

Every permutation $\alpha \in S_n$, $n \geq 2$, is a product of transpositions (Prop. 2.10), so we only need to show that any transposition can be written as a product of adjacencies.
Let $\alpha = \tau_1 \tau_2 \cdots \tau_t$. Let $\tau_i = (i\ j)$, $i < j$. (Note that $\tau_i = (j\ i)$ as well.)
Then $\tau_i = (i\ j) = (i+1\ \ i)(i+2\ \ i+1)\cdots(j-1\ \ j-2)(j-1\ \ j)\cdots(i+1\ \ i+2)(i\ \ i+1)$.
For example, $(1\ 5) = (2\ 1)(3\ 2)(4\ 3)(4\ 5)(3\ 4)(2\ 3)(1\ 2)$.
$\therefore$ α is a product of adjacencies.

(ii) If $i < j$, prove that $(i\ j)$ is a product of an odd number of adjacencies. **Hint**. Use induction on $j - i$.

Proof:

If $j - i = 1$, then $(i\ j)$ is an adjacency. Assume $(i\ j)$ is a product of an odd number of adjacencies for some $j - i > 1$.
Then, by part (i),
$(i\ j) = (i+1\ \ i)(i+2\ \ i+1)\cdots(j-1\ \ j-2)(j-1\ \ j)\cdots(i+1\ \ i+2)(i\ \ i+1)$ is an odd number of adjacencies and
$(i\ \ j+1) = (i+1\ \ i)(i+2\ \ i+1)\cdots(j-1\ \ j-2)\ (j\ \ j-1)\ (j\ \ j+1)\ (j-1\ \ j)\cdots$
$(i+1\ \ i+2)(i\ \ i+1)$.
This is 2 more adjacencies than the odd number of adjacencies whose product is the transposition $(i\ j)$.
Since the sum of an odd number and 2 is odd, then the number of adjacencies whose product is the transposition $(i\ j+1)$ is odd.

2.9 If α is an r-cycle and $1 < k < r$, is α^k an r-cycle?

No.

α is an r-cycle if and only if $\gcd(r, k) = 1$.

Proof:

Let $i_1, i_2, \ldots, i_r$ be distinct integers in $\{1, 2, \ldots n\}$.

$\Rightarrow$:

We will show this direction by contradiction.

Suppose $\gcd(r, k) = d > 1$.

Then $r = dm$ and $k = dn$ for some positive integers m and n. Note that $m > n$.

Thus $\alpha^k(i_1) = \alpha^{dn}(i_1) = i_{dn+1}, \alpha^{dn}(i_{dn+1}) = i_{2dn+1}, \ldots, \alpha^{dn}(i_{(m-1)dn+1}) = i_{dmn+1} = i_1$.

$\therefore$ α^k produces a cycle that does not include i_r, and, hence, cannot be an r-cycle.

$\Leftarrow$:

Now, suppose $\gcd(r, k) = 1$**.**

Then $\forall\ m, n \in \{1, 2, ..., r\}$, such that $m \neq n$, $(mk + 1)(\text{mod } r) \neq (nk + 1)(\text{mod } r)$.

Otherwise, $mk + 1 - rp = nk + 1 - rq$ for some integers p, q.

Thus, $mk - nk = rp - rq = r(p - q)$, forcing $r \mid (m - n)k$.

Since $r \nmid k$, then $r \mid (m - n)$. But $1 \leq m \leq r$, $1 \leq n \leq r$, and $m \neq n$,

so $-r < 1 - n \leq m - n \leq r - n < r$. This gives us that $|m - n| < r$, hence $r \nmid (m - n)$.

$\therefore$ $\{(jk + 1)(\text{mod } r) | j \in \{1, 2, ..., r\}\} = \{1, 2, ..., r\}$.

So then $\forall\ j \in \{0, 1, 2, ..., r\}, \alpha^k(i_{jk+1}) = i_{(j+1)k+1}$.

Thus $\{\alpha^k(i_{jk+1}) \mid j \in \{0, 1, 2, ..., r\}\} = \{i_1, i_2, \ldots, i_r\}$.

$\therefore$ α^k is an r-cycle.

2.11 (i) Prove, for all i, that $\alpha \in S_n$ moves i if and only if α^{-1} moves i.

Proof:

⇒: Assume α moves i. Suppose $\alpha = \beta_1 \beta_2 \cdots (i\ c_1\ c_2 \ldots c_k) \cdots \beta_t$ is the complete factorization of α into disjoint cycles.
Then $\alpha^{-1} = \beta_t^{\,-1} \cdots (c_k \ldots c_1\ i) \cdots \beta_2^{\,-1}\, \beta_1^{\,-1}$ by Proposition 2.4.
By definition of $(c_k \ldots c_1\ i)$, α^{-1} moves i.

⇐: Proof is similar to ⇒ as we can simply interchange α and α^{-1} in the proof above to conclude that if α^{-1} moves i then α moves i.

$\therefore$ For all i, $\alpha \in S_n$ moves i if and only if α^{-1} moves i.

(ii) Prove that if $\alpha, \beta \in S_n$ are disjoint and if $\alpha\beta = (1)$, then $\alpha = (1)$ and $\beta = (1)$.

Proof:

Let $\alpha, \beta \in S_n$ be disjoint permutations such that $\alpha\beta = (1)$,
We will show by contradiction that $\alpha = (1)$ and $\beta = (1)$.
Suppose that $\alpha \neq (1)$.
Then $\exists\ i$ such that $\alpha(i) = j \neq i$. Since α, β are disjoint, then
$\alpha\beta(i) = \alpha(i) = j$.
$\therefore\ \alpha\beta \neq (1)$, a contradiction.
$\therefore\ \alpha = (1)$.
So now we have $\beta = (1)\beta = \alpha\beta = (1)$.
$\therefore\ \alpha = (1)$ and $\beta = (1)$.

2.15 If $n \geq 3$, show that if $\alpha \in S_n$ commutes with every $\beta \in S_n$, then $\alpha = (1)$.

Proof:

Assume $\exists\ \alpha$ such that $\alpha\beta = \beta\alpha$ for every $\beta \in S_n$.
We shall show $\alpha = (1)$ by assuming $\alpha \neq (1)$ and arriving at a contradiction.
Suppose $\alpha \neq (1)$, then $\exists\ i\ , j \in \{1, 2, ..., n\}$ such that $\alpha(i) = j$ and $i \neq j$.
Let $\beta \in S_n$ such that $\beta(i) = i$, $\beta(j) = k$, and $\beta(k) = j$ where $i \neq k \neq j$. Since $n \geq 3$, then we know such a j exists.
So $\alpha\beta(i) = \alpha(i) = j \neq k = \beta(j) = \beta\alpha(i)$, and we have our contradiction.
$\therefore\ \alpha = (1)$.

Thm 2.12
For all $\alpha, \beta \in S_n$, sgn $(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$.

Proof:

Let $a, b \in \{1, 2, \ldots, n\}$ such that $a \neq b$. Let $\tau = (a\ b)$.
We will show

* $\text{sgn}(\tau\alpha) = -\text{sgn}(\alpha)$ for any $\alpha \in S_n$.

Case 1:
Suppose $\alpha = (a\ c_1\ c_2 \ldots c_k)(b\ d_1\ d_2 \ldots d_L)\gamma_1\ \gamma_2 \cdots \gamma_t$, a complete factorization into disjoint cycles where $K, L \geq 0$, $\alpha \in S_n$.
Since $(a\ b)\ (a\ c_1\ c_2 \ldots c_K)(b\ d_1\ d_2 \ldots d_L) = (a\ c_1\ c_2 \ldots c_k\ b\ d_1\ d_2 \ldots d_L)$, then
$\text{sgn}(\tau\alpha) = \text{sgn}((a\ c_1\ c_2 \ldots c_K\ b\ d_1\ d_2 \ldots d_L)\gamma_1\ \gamma_2 \cdots \gamma_t) = (-1)^{n-(t+1)} = -(-1)^{n-(t+2)} =$
$-\text{sgn}((a\ c_1\ c_2 \ldots c_K)(b\ d_1\ d_2 \ldots d_L)\gamma_1\ \gamma_2 \cdots \gamma_t) = -\text{sgn}(\alpha)$.

Case 2:
Suppose $\alpha = (a\ c_1\ c_2 \ldots c_k\ b\ d_1\ d_2 \ldots d_L)\gamma_1\ \gamma_2 \cdots \gamma_t$, a complete factorization into disjoint cycles where $K, L \geq 0$, $\alpha \in S_n$.
Since $(a\ b)\ (a\ c_1\ c_2 \ldots c_K)(b\ d_1\ d_2 \ldots d_L) = (a\ c_1\ c_2 \ldots c_k\ b\ d_1\ d_2 \ldots d_L)$, then
$(a\ b)\ (a\ b)\ (a\ c_1\ c_2 \ldots c_K)(b\ d_1\ d_2 \ldots d_L) = (a\ b)\ (a\ c_1\ c_2 \ldots c_k\ b\ d_1\ d_2 \ldots d_L)$, hence
$(a\ c_1\ c_2 \ldots c_K)(b\ d_1\ d_2 \ldots d_L) = (a\ b)\ (a\ c_1\ c_2 \ldots c_k\ b\ d_1\ d_2 \ldots d_L)$.
$\therefore$ $\text{sgn}(\tau\alpha) = \text{sgn}((a\ c_1\ c_2 \ldots c_K)(b\ d_1\ d_2 \ldots d_L)\gamma_1\ \gamma_2 \cdots \gamma_t) = (-1)^{n-(t+2)} = -(-1)^{n-(t+1)}$
$= -\text{sgn}((a\ c_1\ c_2 \ldots c_K\ b\ d_1\ d_2 \ldots d_L)\gamma_1\ \gamma_2 \cdots \gamma_t) = -\text{sgn}(\alpha)$.

Now suppose $\alpha = \tau_1\ \tau_2 \ldots \tau_m$ where each τ_i is a transposition (apply Prop. 2.10) and $\beta = \gamma_1\ \gamma_2 \ldots \gamma_t$ a complete factorization into disjoint cycles.
We will show sgn $(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$ by induction on m.
Let $m = 1$. Then $\alpha = \tau_1\ (c_1)\ (c_2) \cdots (c_{n-2})$,
hence $\text{sgn}(\alpha) = (-1)^{n-(n-1)} = -1$.
By definition, $\text{sgn}(\beta) = (-1)^{n-t}$.
And by *, $\text{sgn}(\alpha\beta) = -\text{sgn}(\beta) = -(-1)^{n-t}$.
$\therefore$ sgn $(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$.
Let $m > 1$ and assume sgn $(\gamma\beta) = \text{sgn}(\gamma)$ sgn (β) for any γ that is a product of m transpositions.
Define $\alpha = \tau_{m+1}\ \tau_1\ \tau_2 \ldots \tau_m$, where τ_{m+1} is a transposition in S_n.
Let $\gamma = \tau_1\ \tau_2 \ldots \tau_m$.
By *, $\text{sgn}(\alpha) = \text{sgn}(\tau_{m+1}\gamma) = -\text{sgn}(\gamma)$.
So then, by our induction hypothesis,
$\text{sgn}(\alpha\beta) = \text{sgn}\ (\tau_{m+1}\gamma\beta) = -\text{sgn}\ (\gamma\beta) = -\text{sgn}(\gamma)\text{sgn}(\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$.
$\therefore$ sgn $(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$ for all $\alpha, \beta \in S_n$.