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Math 210A
HW #6, Direct Products Homework Handout
#1 – 4

Let G be a group with $H \le G$ and $K \le G$. Define $HK = \{hk| \ h \in H, k \in K\}$.

1. Prove that the following are equivalent.

(a) $HK \le G$.
(b) HK is closed under the binary operation of G restricted to HK.
(c) $HK = KH$.

Proof (a) ⇒(b):
Assume $HK \le G$. Then HK is closed under the binary operation of G restricted to HK by definition of subgroup.

Proof (b) ⇒ (c): Assume HK is closed under the binary operation of G restricted to HK.
We will show $HK \subseteq KH$ and $HK \supseteq KH$.
Let $x \in KH$, then $x = kh$ where $h \in H, k \in K$.
Since $H \le G$ and $K \le G$, then $e \in H$ and $e \in K$.
$\therefore$ $he = h \in HK$ and $ek = k \in HK$. $\therefore$ $x = kh \in HK$ by assumption of closure.
$\therefore$ $KH \subseteq HK$.
For the reverse inclusion, let $y \in HK$. Then $y = hk$ for some $h \in H$ and some $k \in K$.
We know $k^{-1}h^{-1} \in KH$, and since $k^{-1}h^{-1} = (hk)^{-1} = x^{-1}$, then $x^{-1} \in KH$.
But also $k^{-1} = ek^{-1} \in HK$ and $h^{-1} = h^{-1}e \in HK$,
hence $k^{-1}h^{-1} \in HK$ by assumption of closure.
So $x^{-1} = h'k'$ for some $h' \in H$ and $k' \in K$.
Since $x = (x^{-1})^{-1} = (h'k')^{-1} = k'^{-1}h'^{-1} \in KH$, then $HK \supseteq KH$.
$\therefore$ $HK = KH$.

Proof (c) ⇒ (a): Assume $HK = KH$.
We will show HK is nonempty and then apply the subgroup test to show $HK \le G$.
Since $e \in H$ and $e \in K$, then $e = e \bullet e \in HK$. $\therefore$ HK is nonempty.
Let $x, y \in HK$. Then $x = h_1k_1$ for some $h_1 \in H$ and some $k_1 \in K$.
And $y = h_2k_2$ for some $h_2 \in H$ and some $k_2 \in K$.
So $xy^{-1} = h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$.
Let $k_3 = k_1k_2^{-1}$. Clearly $k_3 \in K$.
Since $KH = HK$, then $k_3h_2^{-1} = h_3k_4$ for some $h_3 \in H$ and some $k_4 \in K$.
Let $h_4 = h_1h_3$. Clearly h_4 is in H.
And now, we shall put it all together:
$xy^{-1} = h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1k_3h_2^{-1} = h_1h_3k_4 = h_4k_4 \in \text{HK}$.
$\therefore$ $HK \le G$.

2. Prove that if $H \lhd G$ or $K \lhd G$, then $HK \leq G$.

Proof:
We will show $HK \neq \varnothing$ and then apply the subgroup test.

Since $e \in H$ and $e \in K$, then $e = e \bullet e \in HK$. $\therefore$ HK is nonempty.

Let $x, y \in HK$. Then $x = h_1k_1$ for some $h_1 \in H$ and some $k_1 \in K$.

And $y = h_2k_2$ for some $h_2 \in H$ and some $k_2 \in K$.

So $xy^{-1} = h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$.

Let $k_3 = k_1k_2^{-1}$. Clearly $k_3 \in K$. So $xy^{-1} = h_1k_3h_2^{-1}$.

If $H \lhd G$ then we have $ghg^{-1} \in H$, $\forall\, g \in G, \forall\, h \in H$.

If $K \lhd G$ then we have $gkg^{-1} \in K, \forall\, g \in G, \forall\, k \in K$.

Either way, we have $k_3h_2^{-1} = h_3k_4$ for some $h_3 \in H$ and some $k_4 \in K$.

Let $h_4 = h_1h_3$, also clearly in H.

And now we put it all together.

$xy^{-1} = h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1k_3h_2^{-1} = h_1h_3k_4 = h_4k_4 \in HK$.

$\therefore HK \leq G$.

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3. Prove that if $H \lhd G$ and $K \lhd G$, then $HK \lhd G$.

Proof:
Assume $H \lhd G$ and $K \lhd G$. We have show in Exercise #2 that $HK \leq G$, so we only need to establish normalcy.

Let $g \in G$, $h \in H$, and $k \in K$.

Since $\forall\, g \in G$, ($ghg^{-1} \in H, \forall\, h \in H$, and $gkg^{-1} \in K, \forall\, k \in K$),

then $ghg^{-1}gkg^{-1} = ghkg^{-1} \in HK$. $\therefore HK \lhd G$.

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4. Suppose H and K are finite subgroups of G.
(a) For each $x = hk \in HK$, define ϕ: $H \cap K \to \{(h', k') \in H \times K \mid h'k' = x\}$ by $\phi(d) = (hd, d^{-1}k)$. Prove that ϕ is a bijection.

Proof:
Let $x = hk \in HK$.

We first show ϕ is injective.

Let $m, n \in H \cap K$ such that $\phi(m) = \phi(n)$. Then $(hm, m^{-1}k) = (hn, n^{-1}k)$.

$\therefore$ $hm = hn$ and $m^{-1}k = n^{-1}k$.

Note that $m, n \in H \cap K \Rightarrow m, n \in H$, and $m^{-1}, n^{-1} \in K$.

$\therefore$ We have cancellation for $hm = hn$ which gives us $m = n$.

And we have cancellation for $m^{-1}k = n^{-1}k$, which also gives us $m = n$.

$\therefore$ ϕ is injective.

And now we show ϕ is surjective.

Let $(a, b) \in \{(h', k') \in H \times K \mid h'k' = x\}$. Then $ab = hk$.

Since $h \in G$, $k \in G$, and G is a group, then $b = a^{-1}hk$ and $a = hkb^{-1}$.

Since $a \in H$, then $hkb^{-1} \in H$. And since $h \in H$, then $kb^{-1} \in H$.

But $k \in K$ and $b \in K$, which implies $b^{-1} \in K$, hence $kb^{-1} \in K$.

$\therefore$ $kb^{-1} \in H \cap K$. Similarly $a^{-1}h \in H \cap K$.

Let $d = kb^{-1}$. We will show that $d^{-1} = a^{-1}h$.

Consider $a^{-1}hkb^{-1} = a^{-1}abb^{-1} = 1 = kk^{-1}h^{-1}h = k(hk)^{-1}h = k(ab)^{-1}h = kb^{-1}a^{-1}h$.

$\therefore$ $a^{-1}h = (kb^{-1})^{-1}$ and $\phi(kb^{-1}) = (hkb^{-1}, a^{-1}hk) = (a, b)$, hence, ϕ is surjective.

(b) Prove that if $H \cap K = \{e\}$, then each element of HK can be expressed uniquely as a product hk where $h \in H$ and $k \in K$.

Proof:
Assume $H \cap K = \{e\}$.

Let $x = hk \in HK$.

Suppose $hk = h'k'$ for some $h' \in H$ and $k' \in K$.

Then $h'^{-1}h = k'^{-1}k$. Since $h'^{-1}h \in$ H and $k'^{-1}k \in$ K, then by assumption, $e = h'^{-1}h = k'^{-1}k$.

$\therefore$ $h = h'$ and $k = k'$.

$\therefore$ x can be expressed uniquely as a product hk where $h \in H$ and $k \in K$.

(c) Prove $|HK| = \dfrac{|H||K|}{|H \cap K|}$.

Proof:
Define $f : H \times K \to HK$ by $f(h, k) = hk = x$, clearly a surjection.

Then for each $x \in HK$, we have $f^{-1}(x) = \{(h, k) \in H \times K \mid hk = x\}$.

By part (a), for each $x = hk \in HK$, there is a bijection

ϕ: $H \cap K \to \{(h', k') \in H \times K \mid h'k' = x\}$ defined by $\phi(d) = (hd, d^{-1}k)$.

$\therefore$ For each x, $|f^{-1}(x)| = |H \cap K|$.

And since $H \times K = \bigcup_{x \in HK} f^{-1}(x)$ where each $f^{-1}(x)$ is disjoint, then

$|H \times K| = |\bigcup_{x \in HK} f^{-1}(x)| = |HK| \bullet |f^{-1}(x)|$.

$\therefore$ $|H|\,|K| = |H \times K| = |HK| \bullet |H \cap K|$.

Hence, $|HK| = \dfrac{|H||K|}{|H \cap K|}$.
