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Math 210A
HW #9 The Sylow Theorems #1 – 16

Textbook Problems:

5.23 (i) Prove that if d is a positive divisor of 24, then S_4 has a subgroup of order d.

Proof:

We have the following subgroups of S_4:

1|24	$\{(1)\} \leq S_4$
2|24	$\{(1), (12)\} \leq S_4$
3|24	$\{(1), (123), (132)\} \leq S_4$
4|24	$\{(1), (12)(34), (13)(24), (14)(23)\} \leq S_4$
6|24	$\{(1), (12), (13), (23), (123), (132)\} \leq S_4$
8|24	$\{(1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\} \leq S_4$
12|24	$A_4 \leq S_4$
24|24	$S_4 \leq S_4$

(ii) If $d \neq 4$, prove that any two subgroups of S_4 having order d are isomorphic.

Proof:

Note that

S_4		
No.	Cycle Structure	Order
1	(1)	1
6	(12)	2
8	(123)	3
6	(1234)	4
3	(12)(34)	2

Let H_d denote any subgroup of S_4 having order d.

1|24	$H_1 \cong \{(1)\}$ (there is only 1 isomorphism class of order 1)
2|24	$H_2 \cong \mathbb{Z}_2 \cong \{(1), (12)\}$ (there is only 1 isomorphism class of order 2)
3|24	$H_3 \cong \mathbb{Z}_3 \cong \{(1), (123), (132)\}$ (there is only 1 isomorphism class of order 3)
4|24	$H_4 \cong \boldsymbol{V}_4 \cong \{(1), (12)(34), (13)(24), (14)(23)\}$ or $H_4 \cong \mathbb{Z}_4 \cong \{(1), (13)(42), (1234), (1432)\}$
6|24	$H_6 \cong \{(1), (12), (13), (23), (123), (132)\}$ (A group of order 6 is only isomorphic to S_3 or $\mathbb{Z}_6$. Since S_4 has no element of order 6, then $H_6 \cong S_3$.)
8|24	$H_8 \cong \{(1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$ (Recall Prop 2.58(If G is a group and $g \in G$, then conjugation $\gamma_g : G \to G$ is an isomorphism.) Since H_8 is a Sylow 2-group of S_4 and all Sylow 2-subgroups are conjugates, then all Sylow 2-subgroups are isomorphic.)
12|24	$H_{12} \cong A_4 \leq S_4$ (there is only one subgroup of order 12)
24|24	$H_{24} \cong S_4$ (the orders are the same and $H_{24} \leq S_4$)

5.27 Prove that a Sylow 2-subgroup of A_5 has exactly five conjugates.

Proof:
$|A_5| = 60 = 2^2 \bullet 3 \bullet 5$.

$n_2 \mid 15 \Rightarrow$ $n_2 = 1, 3, 5, 15$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, 7, 9, 11, 13, 15$	$n_2 = 1, 3, 5, 15$
$n_3 \mid 20 \Rightarrow$ $n_3 = 1, 2, 4, 5, 10, 20$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, 10, 13, 17, 20$	$n_3 = 1, 4, 10, 20$
$n_5 \mid 12 \Rightarrow$ $n_5 = 1, 2, 3, 4, 6, 12$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11$	$n_5 = 1, 6$

n_2 = the number of conjugates of a Sylow 2-subgroup of A_5.
Since A_5 is simple, then $n_2 \neq 1$, $n_3 \neq 1$, and $n_5 \neq 1$, hence $n_5 = 6$.
If $n_3 = 4$, then $[G:H_3] = 4$ and $|G| = 60 \nmid 4!$, so $n_3 \neq 4$.
If $n_2 = 3$, then $[G:H_2] = 3$ and $|G| = 60 \nmid 3!$, so $n_2 \neq 3$.
If $n_5 = 6$ and $n_3 = 20$, then A_5 has 6 subgroups of order 5, and 20 subgroups of order 3.
This would require 24 distinct elements of order 5 and 40 distinct elements of order 3. This would exceed the order of A_5, 60.
So $n_3 = 10$.
Now **assume $n_5 = 6$, $n_3 = 10$, and $n_2 = 15$.**
Then A_5 has 6 subgroups of order 5, 10 subgroups of order 3 and 15 subgroups of order 4. Since A_5 has no elements of order 4, then this would require 24 distinct elements of order 5, 20 distinct elements of order 3,

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Deleted: and 45 distinct elements of order 2. Again, we have exceeded order of A_5, 60.

A_5		
No.	Cycle Structure	Order
1	(1)	1
20	(123)	3
24	(12345)	5
15	(12)(34)	2

Notice that elements of order 2 in A_5 are of the cycle structure, (12)(34).. So, for any Sylow 2-subgroups of A_5, only 4 “letters” of {1, 2, …, 5} can be used as using all 5 in one subgroup would create a permutation of cycle structure other that (12)(34). And we can only make 5 such subgroups. $\therefore$ $n_2 \neq 15$.

This leaves only one possibility.
Assume $n_5 = 6$, $n_3 = 10$, and $n_2 = 5$.
Thus A_5 has 6 subgroups of order 5, 10 subgroups of order 3 and 5 subgroups of order 4. This would require 24 distinct elements of order 5, 20 distinct elements of order 3, and 15 elements of order 2. These elements with the identity add up to 60, as desired.
$\therefore$ $n_5 = 10$, which implies that A_5 has exactly five conjugates.

5.28 Prove that there are no simple groups of order 96, 300, 312, or 1000.
Hint. Some of these are not tricky.
Proof:
Let G be group such that **|G| = 96** = $2^5 \bullet 3$.

$n_2 \mid 3 \Rightarrow$ $n_2 = 1, 3$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3$	$n_2 = 1, 3$
$n_5 \mid 32 \Rightarrow$ $n_5 = 1, 2, 4, 8, 16$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16$	$n_5 = 1, 16$

If G is simple, then $n_2 \neq 1$, hence $n_2 = 3$. But $|G| = 96 \nmid 3!$.
So by the Index Factorial theorem, G is not simple, a contradiction to our assumption.
$\therefore$ We have that $n_2 = 1$, hence $\exists\, P_2 \lhd G$ where P_2 is a Sylow 2-subgroup of G.
$\therefore$ G is not simple.

5.28 (cont.) Let G be group such that **|G| = 300** = $2^2 \bullet 3 \bullet 5^2$.

$n_2 \mid 75 \Rightarrow$ $n_2 = 1, 3, 5, 15, 25, 75$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, \ldots, 15, \ldots, 25, \ldots, 75$	$n_2 = 1, 3, 5, 15, 25, 75$
$n_3 \mid 100 \Rightarrow$ $n_3 = 1, 2, 4, 5, 10, 20, 25, 50, 100$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 10, \ldots, 25, \ldots, 100$	$n_3 = 1, 4, 10, 25, 100$
$n_5 \mid 12 \Rightarrow$ $n_5 = 1, 2, 3, 4, 6, 12$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11$	$n_5 = 1, 6$

If G is simple, then $n_5 \neq 1$, hence $n_5 = 6$. But $|G| = 300 \nmid 6!$.
So by the Index Factorial theorem, G is not simple, a contradiction to our assumption.
$\therefore$ We have that $n_5 = 1$, hence $\exists\, P_5 \lhd G$ where P_5 is a Sylow 5-subgroup of G.
$\therefore$ G is not simple.

Let G be group such that **|G| = 312** = $2^3 \bullet 3 \bullet 13$.

$n_2 \mid 39 \Rightarrow$ $n_2 = 1, 3, 13, 39$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, \ldots, 13, \ldots, 39$	$n_2 = 1, 3, 13, 39$
$n_3 \mid 104 \Rightarrow$ $n_3 = 1, 2, 4, 8, 13, 26, 52, 104$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, \ldots, 13, \ldots, 52$	$n_3 = 1, 4, 13, 52$
$n_{13} \mid 24 \Rightarrow$ $n_{13} = 1, 2, 4, 8, 12, 24$	$n_{13} \equiv 1 \pmod{13} \Rightarrow$ $n_{13} = 1, 14, 25$	$n_{13} = 1$

Since $n_{13} = 1$, then $\exists\, P_{13} \lhd G$ where P_{13} is a Sylow 13-subgroup of G.

Let G be group such that **|G| = 1000** = $2^3 \bullet 5^3$.

$n_2 \mid 125 \Rightarrow$ $n_2 = 1, 5, 25, 125$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, \ldots, 5, \ldots, 25, \ldots, 125$	$n_2 = 1, 5, 25, 125$
$n_5 \mid 8 \Rightarrow$ $n_5 = 1, 2, 3, 4, 8$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11$	$n_5 = 1$

Since $n_5 = 1$, then $\exists\, P_5 \lhd G$ where P_5 is a Sylow 5-subgroup of G.

5.29 Let G be a group of order 90.
(i) If a Sylow 5-subgroup P of G is not normal, prove that it has six conjugates.
Hint. If P has 18 conjugates, there are 72 elements in G of order 5. Show that G has more than 18 other elements.
Proof:
Let G be group such that $|G| = 90 = 2 \cdot 3^2 \cdot 5$.

$n_2 \mid 45 \Rightarrow$ $n_2 = 1, 3, 5, 15, 45$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, \ldots, 15, \ldots, 45$	$n_2 = 1, 3, 5, 15, 45$
$n_3 \mid 10 \Rightarrow$ $n_3 = 1, 2, 5, 10$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, 10, \ldots$	$n_3 = 1, 10$
$n_5 \mid 18 \Rightarrow$ $n_5 = 1, 2, 3, 6, 9, 18$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16, \ldots$	$n_5 = 1, 6$

Let P_5 be a Sylow 5-subgroup of G. If P_5 is not normal, then $n_5 \neq 1$, hence $n_5 = 6$.
$\therefore$ P_5 has 6 conjugates in G, (by Sylow theorem, part (2) (All Sylow p-supbgroups are conjugates.))

(ii) Prove that G is not simple.
Hint. Use Exercises 2.95(ii) and 2.96(ii) on page 114.
Proof:
Assume G is simple, then $n_5 \neq 1$, hence $n_5 = 6$. Thus, there are $6 \cdot 4 = 24$ elements of order 5. If $n_3 = 10$, there are 10 subgroups of order 9. If these 10 subgroups intersect trivially, then we have $10 \cdot 8 = 80$ non-identity elements (not of order 5). This is too many elements, 104 for our group of order 90.
So we have Sylow 3-subgroups, P_3 and P_3' such that $|P_3 \cap P_3'| = 3$. Let $Q = P_3 \cap P_3'$.
We know $Q \lhd P_3$ and $Q \lhd P_3'$ as P_3 and P_3' are Abelian by Corollary 2.104 (If p is prime, then every group of order p^2 is Abelian.).
And we know $P_3 \leq N_G(Q)$ and $P_3' \leq N_G(Q)$ as the normalizer is the largest subgroup of G in which Q is normal.
Also, $|P_3| = |P_3'| \neq |N_G(Q)|$ as $P_3 \cup P_3' \subseteq N_G(Q)$ and $P_3 \neq P_3'$.
Let $m = |N_G(Q)|$. And now we have, $|P_3| = |P_3'| \mid |N_G(Q)| \mid |G|$, hence $9 \mid m \mid 90$.
So, m = 18, 45, or 90.
If $m = 18$, then $[G: N_G(Q)] = 90/18 = 5$. But we have assumed G is simple, yet $90 \nmid 5!$, a contradiction, by the Index Factorial theorem. $\therefore$ $m \neq 18$.
If $m = 45$, then $[G: N_G(Q)] = 90/45 = 2$, hence $N_G(Q) \lhd$ G, another contradiction.
If $m = 90$, then $[G: N_G(Q)] = 90/90 = 1$, hence $G = N_G(Q)$ which means Q is normal in G.
So there is no escaping a contradiction to our assumption that G is simple, thus G is not simple.

5.30 Prove that there is no simple group of order 120.

Proof:
Let G be group such that $|G| = 120 = 2^3 \bullet 3 \bullet 5$.

$n_2 \mid 15 \Rightarrow$ $n_2 = 1, 3, 5, 15$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, 15, \ldots$	$n_2 = 1, 3, 5, 15$
$n_3 \mid 40 \Rightarrow$ $n_3 = 1, 2, 4, 5, 8, 10, 20, 40$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, 10, \ldots, 40$	$n_3 = 1, 4, 7, 10, 40$
$n_5 \mid 24 \Rightarrow$ $n_5 = 1, 2, 3, 6, 12, 24$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16, \ldots$	$n_5 = 1, 6$

Assume G is simple. Then $n_2 \neq 1$, $n_3 \neq 1$, and $n_5 \neq 1$, hence $n_5 = 6$.

Then, by Representation on Cosets, $\exists\ \phi : G \to S_6$ where $\ker \phi \leq N_G(P_5)$.

Since G is simple by assumption, then $\ker \phi = \{e\}$. So $G \cong \phi(G) \leq S_6$.

Notice that $\phi(G) \cap A_6 \lhd \phi(G)$ (by 2nd Isomorphism theorem).

And by Exam 1, # 7(c), $|\phi(G) \cap A_6| = |\phi(G)| = 120$ or $|\phi(G) \cap A_6| = (1/2)|\phi(G)| = 60$.

If $\sim(\phi(G) \leq A_6)$, then $\phi(G) \cap A_6 \neq \phi(G)$, hence $|\phi(G) \cap A_6| = (1/2)|\phi(G)| = 60$.

But since $\phi(G) \cap A_6 \lhd \phi(G)$, $\phi(G)$ is simple by assumption, and $|\phi(G) \cap A_6| = 60$, then we have a contradiction.

If $\phi(G) \leq A_6$, then $\phi(G) \cap A_6 = \phi(G)$, hence $|\phi(G) \cap A_6| = |\phi(G)| = 120$.

So $[A_6 : \phi(G) \cap A_6] = 360/120 = 3$.

Then by the Representation on Cosets theorem,

$\exists\ \psi : A_6 \to S_3$ where $\ker \psi \leq A_6/\phi(G) \cap A_6$. And since A_6 is simple, then $\ker \psi = \{(1)\}$.

Thus, by the 1st Isomorphism theorem, $A_6 \cong \psi(A_6)$.

But $\psi(A_6) \leq S_3$ and $|A_6| = 120 > 6 = |S_3|$.
So $\ker \phi \neq \{e\}$, hence G is not simple.

5.31 Prove that there is no simple group of order 150.

Proof:
Let G be group such that $|G| = 150 = 2 \bullet 3 \bullet 5^2$.

$n_2 \mid 45 \Rightarrow$ $n_2 = 1, 3, 5, 9, 15, 45$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, 7, \ldots$	$n_2 = 1, 3, 5, 9, 15, 45$
$n_3 \mid 30 \Rightarrow$ $n_3 = 1, 2, 3, 5, 6, 10, 15, 30$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, \ldots, 10, \ldots$	$n_3 = 1, 10$
$n_5 \mid 6 \Rightarrow$ $n_5 = 1, 2, 3, 6$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16, \ldots$	$n_5 = 1, 6$

If G is simple, then $n_5 \neq 1$, hence $n_5 = 6$. But $|G| = 150 \nmid 6!$.
So by the Index Factorial theorem, G is not simple, a contradiction to our assumption.
$\therefore$ We have that $n_5 = 1$, hence $\exists\ P_5 \lhd G$ where P_5 is a Sylow 5-subgroup of G.

$\therefore$ G is not simple.

Worksheet Problems:
2. Prove the following:
(a) Every group of order 15 is Abelian.

Proof:
Let G be group such that $|G| = 15 = 3 \bullet 5$.

$n_3 \mid 5 \Rightarrow$ $n_3 = 1, 5$	$n_3 \equiv 1 \pmod{3} \Rightarrow$ $n_3 = 1, 4, \ldots$	$n_3 = 1$
$n_5 \mid 3 \Rightarrow$ $n_5 = 1, 3$	$n_5 \equiv 1 \pmod{5} \Rightarrow$ $n_5 = 1, 6, \ldots$	$n_5 = 1$

Since $n_3 = 1$, and $n_5 = 1$, then $\exists\ P_3 \triangleleft G$ and $\exists\ P_5 \triangleleft G$ where P_3 is a Sylow 3-subgroup of G and P_5 is a Sylow 5-subgroup of G.

Since $|P_3| = 3$ and $|P_5| = 5$, then by Corollary 2.45 (Every group of prime order is cyclic.) $\exists\ a, b \in G$ such that $\circ(a) = 3$, $\circ(b) = 5$, $\langle a \rangle = P_3$, and $\langle b \rangle = P_5$.

Consider $\langle a \rangle \langle b \rangle$.

We know $\langle a \rangle \langle b \rangle \leq G$. And $(|\langle a \rangle|, |\langle b \rangle|) = 1$, hence $|\langle a \rangle \langle b \rangle| = 15 = |G|$. $\therefore$ $\langle a \rangle \langle b \rangle = G$.

Also, $\langle a \rangle \cap \langle b \rangle = \{e\}$. And so G is the internal direct product of $\langle a \rangle$ and $\langle b \rangle$.

And by Direct Products Theorem 4 (If G is the internal direct product of H and K, then $HK \cong H \times K$.), $G \cong \langle a \rangle \times \langle b \rangle$. But $\langle a \rangle \cong \mathbb{Z}_3$ and $\langle b \rangle \cong \mathbb{Z}_5$ so $G \cong \mathbb{Z}_3 \times \mathbb{Z}_5$. And by Finite Abelian Groups, Exercise 1, ($\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ if and only if $(n, m) = 1$.), we have $G \cong \mathbb{Z}_{15}$.

Since $\mathbb{Z}_{15}$ is cyclic, then G is cyclic, hence Abelian.

2. (b) There are no more than 4 non-isomorphic groups of order 30.

Proof:
Let G be group such that $|G| = 30 = 2 \bullet 3 \bullet 5$.

$n_2 \mid 15 \Rightarrow$ $n_2 = 1, 3, 5, 15$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, 15, \ldots$	$n_2 = 1, 3, 5, 15$
$n_3 \mid 10 \Rightarrow$ $n_3 = 1, 2, 5, 10$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, 10, \ldots$	$n_3 = 1, 10$
$n_5 \mid 6 \Rightarrow$ $n_5 = 1, 2, 3, 6$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16, \ldots$	$n_5 = 1, 6$

If $n_3 = 10$ and $n_5 = 9$, then we need 24 distinct elements of order 5 and 20 distinct elements of order 3, a contradiction to $|G| = 30$.
$\therefore$ Either $n_3 = 1$ or $n_5 = 1$. If $n_3 = 1$, then $\exists\, a \in G$ such that $\circ(a) = 3$, $\langle a \rangle = P_3$ and $\langle a \rangle \lhd G$.
And since and $n_5 = 1, 6$, we also have that $\exists\, b \in G$ such that $\circ(b) = 5$, and $\langle b \rangle = P_5$.
By Direct Products, Exercise 2 ($H \le G, K \le G, H \lhd G$ or $K \lhd G \Rightarrow HK \le G$), we have that $\langle a \rangle\langle b \rangle \le G$ and $\langle a \rangle\langle b \rangle \cong \mathbb{Z}_{15}$.by part (a). Thus, $\exists\, c \in G$ such that $\circ(c) = 15$ and $\langle c \rangle \le G$.
By similar argument, if $n_5 = 1$ and $n_3 = 1, 10$ we have the same result.
Since $[G: \langle c \rangle] = 2$, then $\langle c \rangle \lhd G$. So by Cauchy's theorem, $\exists\, d \in G$ such that $\circ(d) = 2$.
So $\langle c \rangle \lhd G$ gives us that $d\langle c \rangle d^{-1} \in \langle c \rangle$. Hence $dcd^{-1} = c^n$ where $0 \le n \le 14$.
Thus $c = d^{-1}c^n d = dc^n d^{-1} = (dcd^{-1})^n = (c^n)^n = c^{n^2}$. $\therefore$ $e = c^{n^2-1}$. By corollary to Lagrange's theorem, $15 \mid n^2 - 1$. So $n^2 - 1 = 1, 4, 11$, or 14. This gives us the following possible groups isomorphic to G:
$\langle c, d: c^{15} = 1 = d^2; cdc^{-1} = d \rangle$, $\langle c, d: c^{15} = 1 = d^2; cdc^{-1} = d^4 \rangle$, $\langle c, d: c^{15} = 1 = d^2; cdc^{-1} = d^{11} \rangle$, $\langle c, d: c^{15} = 1 = d^2; cdc^{-1} = d^{14} \rangle$.
Thus, there are no more than 4 non-isomorphic groups of order 30.

(c) There are at least 4 non-isomorphic groups of order 30. (Describe them in terms of groups that we know and explain how you know that the four you've described are non-isomorphic.

Proof:
$|\mathbb{Z}_{30}| = |D_{30}| = |D_{10} \times \mathbb{Z}_3| = |D_6 \times \mathbb{Z}_5| = 30$.
$\mathbb{Z}_{30}$ has an element of order 30.
D_{30} is not cyclic and has 15 elements of order 2.
$D_{10} \times \mathbb{Z}_3$ has only 1 element of order 2 and 5 elements of order 6.
$D_6 \times \mathbb{Z}_5$ has only 1 element of order 6 and only 1 element of order 2.
$\therefore$ None of the 4 groups have the same number of elements of the same order, hence none of them are isomorphic to each other.
$\therefore$ There are at least 4 non-isomorphic groups of order 30.

3. Let G be a group of order 48. Show that the intersection of any two distinct Sylow 2-subgroups of G has order 8.

Proof:
$48 = 2^4 \bullet 3$.

$n_2 \mid 3 \Rightarrow$ $n_2 = 1, 3$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, \ldots$	$n_2 = 1, 3$
$n_3 \mid 16 \Rightarrow$ $n_3 = 1, 2, 4, 8, 16$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, \ldots, 16, \ldots$	$n_3 = 1, 4, 16$

By hypothesis, we assume $n_2 = 3$ (i.e. $\exists\ P_2, P_2'$, and P_2'', distinct Sylow 2-subgroups of G, each of order 16.).

Note that $|P_2 \cap P_2'| \neq 16$ as the subgroups are distinct, hence $|P_2 \cap P_2'| = 1, 2, 4,$ or 8.

Since $n_2 = [G:N_G(P_2)] = 3 = [G:P_2]$, then $N_G(P_2) = P_2$.

By the Representation of Cosets theorem, $\exists\ \phi:G \to S_3$ such that $\ker \phi \leq P_2$.

Since $|G| = 48$ and $|S_3| = 6$, then the map is an 8 to 1 map, hence $|\ker \phi| \geq 8$.

We know $\sim(P_2 \lhd G)$ but $\ker \phi \lhd$ G, so $\ker \phi \neq P_2$.

We also know P_2 is conjugate to P_2', so $\exists\ g \in G$ such that $P_2' = gP_2g^{-1}$.

Since $\ker \phi \lhd G$, then $g(\ker \phi)g^{-1} = \ker \phi$.

And, since $\ker \phi \leq P_2$, $\ker \phi = g(\ker \phi)g^{-1} \subseteq gP_2g^{-1} = P_2'$.

$\therefore$ $\ker \phi \subseteq P_2 \cap P_2'$.

Since $|\ker \phi| \geq 8$, $\ker \phi \subseteq P_2 \cap P_2'$, and $|P_2 \cap P_2'| \leq 8$, then $|P_2 \cap P_2'| = 8$.

$\therefore$ The intersection of any two distinct Sylow 2-subgroups of G has order 8.

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4. Let G be a group with $|G| = 56$. Prove that G is not simple.

Proof:
$56 = 2^3 \bullet 7$.

$n_2 \mid 7 \Rightarrow$ $n_2 = 1, 7$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, 7, \ldots$	$n_2 = 1, 7$
$n_7 \mid 8 \Rightarrow$ $n_7 = 1, 2, 4, 8$	$n_7 \equiv 1 \pmod 7 \Rightarrow$ $n_7 = 1, 4, 7, \ldots$	$n_7 = 1, 8$

Assume G is simple, then $n_2 \neq 1$ and $n_7 \neq 1$. Thus, $n_2 = 7$ and $n_7 = 8$, which gives us 7 Sylow 2-subgroups each of order 8 and 8 Sylow 7-subgroups each of order 7.

Since 7 is prime, then $7 \bullet 6 = 42$ elements of our group of order 56 have order 7.

The 7 subgroups of order 8 have intersections of 1, 2, or 4 elements, hence we have a minimum of $5 \bullet 4 = 20$ elements of order $\neq 7$, thus exceeding the size of our group.

$\therefore$ $n_2 = 1$ or $n_7 = 1$, hence G is not simple.

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5. What is the smallest composite integer n such that there is a unique group of order n?
15.

Proof:

Any group with order an even composite integer is isomorphic to a dihedral group as well as a cyclic group. So n must be odd. The smallest odd composite integer is 15.
By Exercise 1 (a) above, any group of order 15 is cyclic and Abelian, hence unique.

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6. Let G be a noncyclic group of order 21.
(a) How many 3-Sylow subgroups does G have? 7.

Proof:

$n_3 \mid 7 \Rightarrow$ $n_3 = 1, 7$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, \ldots$	$n_3 = 1, 7$
$n_7 \mid 3 \Rightarrow$ $n_7 = 1, 3$	$n_7 \equiv 1 \pmod 7 \Rightarrow$ $n_7 = 1, 4, \ldots$	$n_7 = 1$

If $n_3 = 1$, and $n_7 = 1$, then $\exists\, P_3 \lhd G$ and $\exists\, P_7 \lhd G$ where P_3 is a Sylow 3-subgroup of G and P_7 is a Sylow 7-subgroup of G.

Since $|P_3| = 3$ and $|P_7| = 7$, then

$\exists\, a, b \in G$ such that $\circ(a) = 3$, $\circ(b) = 7$, $\langle a \rangle = P_3$, and $\langle b \rangle = P_7$.

We know $\langle a \rangle\langle b \rangle \leq G$. And $(|\langle a \rangle|, |\langle b \rangle|) = 1$, hence $|\langle a \rangle\langle b \rangle| = 21 = |G|$. $\therefore$ $\langle a \rangle\langle b \rangle = G$.

Also, $\langle a \rangle \cap \langle b \rangle = \{e\}$. And so G is the internal direct product of $\langle a \rangle$ and $\langle b \rangle$.

Thus, $G \cong \langle a \rangle \times \langle b \rangle$. But $\langle a \rangle \cong \mathbb{Z}_3$ and $\langle b \rangle \cong \mathbb{Z}_7$ so $G \cong \mathbb{Z}_3 \times \mathbb{Z}_7$. And so we have $G \cong \mathbb{Z}_{21}$, which is cyclic.

$\therefore$ $n_3 = 7$, hence G has 7 Sylow 3-subgroups.

(b) Prove that G has 14 elements of order 3.

Proof:

Since G has 7 distinct Sylow 3-subgroups, then G has $7 \bullet 2$ distinct elements of order 3.

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7. Let G be a group of order 60. Show that G has exactly four elements of order 5 or exactly 24 elements of order 5. Which of these cases holds for A_5?

Proof:
$60 = 2^2 \bullet 3 \bullet 5$

$n_2 \vert 15 \Rightarrow$ $n_2 = 1, 3, 5, 15$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, 15, \ldots$	$n_2 = 1, 3, 5, 15$
$n_3 \vert 20 \Rightarrow$ $n_3 = 1, 2, 4, 5, 10, 20$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, 10, \ldots$	$n_3 = 1, 4, 10$
$n_5 \vert 12 \Rightarrow$ $n_5 = 1, 2, 3, 4, 6, 12$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16, \ldots$	$n_5 = 1, 6$

If $n_5 = 1$, then there is a unique Sylow 5-subgroup of G such that $|P_5| = 5$, hence G has exactly four elements of order 5.
If $n_5 = 6$, then there are 6 unique Sylow 5-subgroups of G, each of order 5.
Since the subgroups are unique and of prime order, the intersection of any 2 of them has order 1. $\therefore$ In this case, G has exactly $6 \bullet 4 = 24$ elements of order 5.

Since A_5 is simple, then $n_5 \neq 1$, hence A_5 has exactly 24 elements of order 5.

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8. Let G be a group of order 60 and let $H \lhd G$ with $|H| = 2$. Show
(a) G has normal subgroups of order 6, 10, and 30,

Proof:
Since $H \lhd G$ with $|H| = 2$, then G/H is a group and $|G/H| = 30$.
And by Exercise 2 (b) $\exists\ S^* \leq G/H$ such that S^* is cyclic, $S^* \lhd G/H$, and $|S^*| = 15$.
By Exercise 2 (a) $\exists\ T^* \leq S^*/H$ such that T^* is cyclic, $T^* \lhd G/H$, and $|T^*| = 5$, and
$\exists\ U^* \leq S^*/H$ such that U^* is cyclic, $U^* \lhd G/H$, and $|U^*| = 3$,
So, by the Correspondence theorem,
$\exists\ S \leq G$, $T \leq G$, and $U \leq G$ such that
$S^* = S/H$, $S \lhd G$, $|S| = 30$,
$T^* = T/H$, $T \lhd G$, $|T| = 10$, and
$S^* = U/H$, $U \lhd G$, $|U| = 6$

(b) G has subgroups of order 12 and 20, and

Proof:
Since $|G/H| = 30$, then $\exists\ P_3^*, P_5^*$ both normal to G/H, and $\exists\ P_2^* \leq G/H$.
So $P_2^*P_3^* \leq G/H$, and $P_2^*P_5^* \leq G/H$, where $|P_2^*P_3^*| = 6$ and $|P_2^*P_5^*| = 10$.
By the Correspondence theorem $\exists\ H_{12}$ and H_{20}, subgroups of G, such that $H_{12} = P_2^*P_3^*/\mathrm{H}$, $H_{20} = P_2^*P_5^*/\mathrm{H}$, $|H_{12}| = 12$, and $|H_{20}| = 20$.

(c) G has a cyclic subgroup of order 30.
Proof:
(stuck)

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9. Let G be a group of order 60. If the Sylow 3-subgroup is normal, show that the Sylow 5-subgroup is also normal.
Proof:
(stuck)

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10. Let $|G| = 7^2 \bullet 13$. Prove G is Abelian.

Proof:

$n_7 \mid 13 \Rightarrow$ $n_7 = 1, 13$	$n_7 \equiv 1 \pmod{7} \Rightarrow$ $n_7 = 1, 4, 7, \ldots$	$n_7 = 1$
$n_{13} \mid 49 \Rightarrow$ $n_{13} = 1, 7, 49$	$n_{13} \equiv 1 \pmod{13} \Rightarrow$ $n_{13} = 1, 12, \ldots$	$n_{13} = 1$

Since $n_7 = 1$ and $n_{13} = 1$, then by Proposition 5.39 (A finite group G all of whose Sylow subgroups are normal is the direct product of its Sylow subgroups.), $G = P_7 \times P_{13}$. Since $|P_7| = 49$, then P_7 is Abelian. $\therefore$ $P_7 \cong \mathbb{Z}_{49}$ or $P_7 \cong \mathbb{Z}_7 \times \mathbb{Z}_7$. And since 13 is prime, $P_{13} \cong \mathbb{Z}_{13}$. Thus, $G \cong \mathbb{Z}_{49} \times \mathbb{Z}_{13}$, or $G \cong \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{13}$, hence G is Abelian.

A group is said to be solvable if there exist subgroups $G_0, G_1, \ldots, G_k$ such that

$$\{e\} = G_k \lhd G_{k-1} \lhd \cdots \lhd G_2 \lhd G_1 \lhd G_0 = G$$

and such that G_i/G_{i+1} is Abelian for all i. This sequence of subgroups is called a solvable series for G.

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11. Prove that S_4 is solvable.

Proof:
$\{(1)\} \lhd \boldsymbol{V}_4 \lhd A_4 \lhd S_4$.
$[S_4 : A_4] = 2$. $\therefore$ S_4/A_4 is Abelian since 2 is prime.
$[A_4 : \boldsymbol{V}_4] = 3$. $\therefore$ $A_4 : \boldsymbol{V}_4$ is Abelian since 3 is prime.
$\boldsymbol{V}_4/\{(1)\} = \boldsymbol{V}_4$ and we know $\boldsymbol{V}_4$ is Abelian.
$\therefore$ S_4 is solvable.

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12. Prove that if G is solvable and $H \leq G$, then H is solvable.

Proof:
G is solvable $\Rightarrow$ exist subgroups $G_0, G_1, \ldots, G_k$ such that
$\{e\} = G_k \lhd G_{k-1} \lhd \cdots \lhd G_2 \lhd G_1 \lhd G_0 = G$.
Let $H_0 = H$ and $H_i = H \cap G_i$. Then by the 2nd Isomorphism theorem, $H \cap G_i \lhd H$.
(stuck)

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13. Suppose G is solvable and $\phi: G \to \overline{G}$ is a homomorphism from G to $\overline{G}$. Prove $\phi(G)$ is solvable.

Proof:
G is solvable $\Rightarrow$ exist subgroups $G_0, G_1, \ldots, G_k$ such that
$\{e\} = G_k \lhd G_{k-1} \lhd \cdots \lhd G_2 \lhd G_1 \lhd G_0 = G$.
Suppose ker $\phi = \{e\}$, then by the Correspondence theorem,
$\phi(G_i) \lhd \phi(G_{i-1})$ for every $i \in \{0, 1, \ldots, k\}$ and $\phi(G_0) = \phi(\overline{G})$.
Suppose G_j = ker ϕ for some $j \in \{0, 1, \ldots, k\}$, then by the Correspondence theorem,
$\phi(G_i) \lhd \phi(G_{i-1})$ for every $i \in \{0, 1, \ldots, j\}$, $\phi(G_0) = \phi(\overline{G})$, and
$\phi(G_m) \subseteq \{e_{\overline{G}}\}$ for every $m \in \{j+1, \ldots, k\}$, hence $\phi(G_m) \lhd \phi(G_{m-1})$.
Need to show $\phi(G_{i-1})/\phi(G_i)$ is Abelian for every i.
(stuck)
$\therefore$ $\overline{G}$ is solvable.

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14. Let G be a group with $H \lhd G$. Suppose H and G/H are both solvable. Prove G is solvable.

Proof:
H is solvable $\Rightarrow$ exist subgroups $H_0, H_1, \ldots, H_k$ of H such that
$\{e\} = H_k \lhd H_{k-1} \lhd \cdots \lhd H_2 \lhd H_1 \lhd H_0 = H$.
G/H is solvable $\Rightarrow$ exist subgroups $N_0, N_1, \ldots, N_k$ of G/H such that
$\{e\} = N_k \lhd N_{k-1} \lhd \cdots \lhd N_2 \lhd N_1 \lhd N_0 = N$.
By the Correspondence theorem, there are subgroups P_i of G with $H \leq P_i$
such that $P_i / H = N_i$ and $P_i \lhd P_{i-1}$. So $H \lhd P_k/H \lhd P_{k-1}/H \lhd \cdots \lhd P_1/H \lhd P_0/H = G/H$.
By the 3rd Isomorphism theorem, $P_i/P_{i-1} \cong (P_i/H)/(P_{i-1}/H)$. And since $(P_i/H)/(P_{i-1}/H)$ is Abelian, then P_i/P_{i-1} is Abelian.
$\therefore$ $\{e\} = H_k \lhd H_{k-1} \lhd \cdots \lhd H_2 \lhd H_1 \lhd H_0 = H \lhd P_k \lhd P_{k-1} \lhd \cdots \lhd P_1 \lhd P_0 = G$.
$\therefore$ G is solvable.

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15. Let G be a group with $H \leq G$ and $K \lhd G$. Prove that if H and K are both solvable, then HK is solvable.
Proof:

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16. Let G be a group of order $495 = 3^2 \bullet 5 \bullet 11$.
(a) What are the possible numbers of Sylow subgroups? (ie. what are the possibilities for n_3, n_5, and n_{11}?)

Proof:

$n_3 \mid 55 \Rightarrow$ $n_3 = 1, 5, 11, 55$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, \ldots, 55, \ldots$	$n_3 = 1, 55$
$n_5 \mid 99 \Rightarrow$ $n_5 = 1, 3, 9, 11, 33, 99$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16, \ldots$	$n_5 = 1, 11$
$n_{11} \mid 45 \Rightarrow$ $n_{11} = 1, 3, 5, 9, 15, 45$	$n_{11} \equiv 1 \pmod{11} \Rightarrow$ $n_{11} = 1, 10, \ldots, 45, \ldots$	$n_{11} = 1, 45$

(b) Prove that a 5-Sylow subgroup or an 11-Sylow subgroup is normal.

Proof:
If $n_5 = 11$ and $n_{11} = 45$, then there are 11 distinct subgroups of order 5 and 45 distinct subgroups of order 11. Thus, G contains $11 \bullet 4 = 44$ elements of order 5 and $45 \bullet 10 = 450$ elements of order 11. We also have at least one subgroup of order 9 which contains 8 elements of order 3 or 9. So G contain $44 + 450 + 8 = 502$ elements, a contradiction to $|G| = 495$. $\therefore$ $n_5 = 11$ or $n_{11} = 45$, thus a 5-Sylow subgroup or an 11-Sylow subgroup is normal in G.

(c) Let K be the normal subgroup from part (b). Prove G/K is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_m \times \mathbb{Z}_9$ where $m \in \{5, 11\}$.

Proof:
Let M be the other subgroup from part (b) and let P_3 be a Sylow 3-subgroup of G.
Suppose $m = 5$, then $|G/K| = 3^2 \bullet 5 \bullet 11 = 5 = 3^2 \bullet 11$. So $n_3 = 1$ and $n_{11} = 1$.
Thus by Proposition 5.39 (A finite group G all of whose Sylow subgroups are normal is the direct product of its Sylow subgroups.), $G/K \cong P_3P_{11}$. Since $|P_3| = 9$, then P_3 is Abelian. $\therefore$ $P_3 \cong \mathbb{Z}_9$ or $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. And since 11 is prime, $P_{11} \cong \mathbb{Z}_{11}$.
Thus, $G \cong \mathbb{Z}_{11} \times \mathbb{Z}_9$, or $G \cong \mathbb{Z}_{11} \times \mathbb{Z}_3 \times \mathbb{Z}_3$.
If $m = 11$, then by similar proof we have $G \cong \mathbb{Z}_5 \times \mathbb{Z}_9$, or $G \cong \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

(d) Let H_5 be a 5-Sylow subgroup of G and let H_{11} be an 11-Sylow subgroup of G.
Prove $H_5H_{11} \triangleleft G$.
(part (c) might be helpful – one of these two subgroups is the K in part (c)).

Proof:
By the Second Isomorphism theorem, $H_5H_{11}/H_5 \cong H_{11}/H_5 \cap H_{11}$. Since $H_5 \cap H_{11} = \{e\}$, then $H_5H_{11}/H_5 \cong H_{11}$. And since $|H_{11}| = 11$, then $H_{11} \cong \mathbb{Z}_{11}$.
Since, by part (c) G/H_5 is isomorphic to $\mathbb{Z}_{11} \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_{11} \times \mathbb{Z}_9$, and $\mathbb{Z}_{11} \triangleleft \mathbb{Z}_{11} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_{11} \triangleleft \mathbb{Z}_{11} \times \mathbb{Z}_9$, then by the Correspondence theorem, $H_5H_{11} \triangleleft G$.

(e) Find a solvable series for G.
Proof:
$\{e\} \triangleleft H_5 \triangleleft H_5H_{11} \triangleleft G$.
