Debra Griffin
Math 210A
HW #12 Ch 3.8 #81, 82, 83 + Worksheet #2, 3, 4

Textbook Problems:

3.81 Prove that if $I = \{0\}$, then $R/I \cong R$.
Proof:
Let π: $R/I \to R$ be defined by $\pi(r + \{0\}) = r$. Since
$\pi(r + \{0\} + s + \{0\}) = \pi(r + s + \{0\}) = r + s = \pi(r + \{0\}) + \pi(s+ \{0\})$ and
$\pi((r + \{0\})(s + \{0\}) = \pi(rs + \{0\}) = rs = \pi(r + \{0\})\pi(s+ \{0\})$, then
π is a homomorphism.
π is surjective since $\forall\ r \in R$, $\pi(r + \{0\}) = r$.
π is injective as $\pi(r + \{0\}) = \pi(s + \{0\})$
$\Rightarrow r = s$
$\Rightarrow r - s = 0$
$\Rightarrow r - s \in \{0\}$
$\Rightarrow r + \{0\} = s + \{0\}$.
$\therefore$ π: $R/I \to R$ is an isomorphism.

2nd proof:
Let i:$R \to R$ be the identity map. Since $i(r + s) = r + s = i(r) + i(s)$ and $i(rs) = rs = i(r)i(s)$, then i is a homomorphism. We know i is surjective as $r \in R$, $i(r) = r$.
Since $\{0\} = \ker f$, then by the first isomorphism theorem, $R/I \cong R$.

3.82 (Third Isomorphism Theorem for Rings) If R is a commutative ring having ideals $I \subseteq J$, then J/I is an ideal in R/I and there is a ring isomorphism $(R/I)/(J/I) \cong R/J$.

Proof:
We first show J/I is an ideal in R/I.
Let $a \in J/I$ and $b \in R/I$. Then $a = j + I$ for some $j \in J$ and $b = r + I$ for some $r \in R$.
Since $j \in J \subseteq R$, then $jr \in R$, hence $ab = (j + I)(r + I) = jr + I \in R/I$.
Thus, J/I is an ideal in R/I.
We next will show $(R/I)/(J/I) \cong R/J$.
Define $f : R/I \to R/J$ by $f(r + I) = r + J$.
To verify f is well-defined, let $c, d \in R/I$ such that $c = d$.
Then $c = r + I$ and $d = s + I$ for some $r, s \in R$.
Thus, $r - s \in I \subseteq J$, so $r - s \in J$.
Hence $r + J = f(r + I) = f(s + I) = s + J$, and we have f is well-defined.
We verify surjectivity of f by letting $r + J \in R/J$ and noting that $f(r + J) = r + I$.
The kernel of $f = \{r + I \in R/I \mid f(r + I) = r + J = J\}$.
Since $r + J = J \Leftrightarrow r \in J$, then $\ker f = J/I$.
So, by the first isomorphism theorem, $(R/I)/(J/I) \cong R/J$.

3.83 For every commutative ring R, prove that $R[x]/(x) \cong R$.

Proof:
Let R be a commutative ring. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n \in R$.
Define φ: $R[x] \to R$ by $\varphi(f(x)) = a_0$.
We know φ is surjective as $\forall\, r \in R$, $\varphi(r) = r$, where r is a polynomial of degree 0.
We will now check that φ is a ring homomorphism.
Let $f, g \in R[x]$. Then $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for some $a_0, a_1, \ldots, a_n \in R$, and $g = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$ for some $b_0, b_1, \ldots, b_n \in R$.
Since R is commutative, we have
$\varphi(f + g) = \varphi(a_0 + b_0 + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n) = a_0 + b_0 = \varphi(f) + \varphi(g)$.
And $\varphi(fg) = \varphi(\sum_{i=0}^{n}\left(\sum_{j+k=i} a_jb_k\right)x^i) = a_0b_0 = \varphi(f)\varphi(g)$.
Note that $\ker \varphi = \{f \in R[x] \mid \varphi(f(x)) = \varphi(a_0 + a_1x + \cdots + a_nx^n) = a_0 = 0_R\}$.
Let $g \in (x)$, then $g = xf$ for some $f \in R[x]$, and $xf = x(a_0 + a_1x + \cdots + a_nx^n)$.
Relabeling the coefficients, we have
$x(a_0 + a_1x + \cdots + a_nx^n) = b_0 + b_1x + \cdots + b_nx^n$ where $b_0 = 0$. $\therefore$ $g \in \ker \varphi$.
Let $f \in \ker \varphi$, then $\varphi(f(x)) = \varphi(a_0 + a_1x + \cdots + a_nx^n) = a_0 = 0_R$.
Since $a_0 = 0_R$, then $f(x) = 0 + x(a_1 + a_2x + \cdots + a_nx^n) = x(g(x))$ for some $g \in R[x]$, hence $f \in (x)$.
$\therefore$ $\ker \varphi = (x)$.
By the first isomorphism theorem, $R[x]/(x) \cong R$.

Worksheet Problems

2. Let F be a field of order 8.
(a) Prove that char $(F) = 2$.

Proof:
Let $a \in F$ with order k.
Note that $(1 + 1)(1 + 1)(1 + 1) = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 0$.
$\therefore$ $1 + 1 = 0$, as F has no zero divisors. Hence $\circ(1) = 2$. Thus $a(1 + 1) = 2a = 0$.
And this gives us that char $(F) = 2$.

(b) Suppose $a, b \in F$ with $a^2 + ab + b^2 = 0$.
(i) Prove $a^3 = b^3$.

Proof:
Since $a^3 - b^3 = (a - b)(a^2 + ab + b^2) = (a - b) \bullet 0 = 0$, then $a^3 = b^3$.

(ii) Prove $a^2 = b^2$.

Proof:
If $a = 0$ or $b = 0$, then $a^2 = b^2$.
If $a \neq 0$ and $b \neq 0$, then $a, b \in U(F)$.
And since $U(F)$ is a multiplicative group of order 7, then $a^7 = b^7 = 1$.
Note that $a^3 = b^3 \Rightarrow a^9 = a^3 \bullet a^3 \bullet a^3 = b^3 \bullet b^3 \bullet b^3 = b^9$.
Thus, $a^2 = a^2 \bullet 1 = a^2 \bullet a^7 = a^9 = b^9 = b^7 \bullet b^2 = 1 \bullet b^2 = b^2$, as desired.

(iii) Prove $a = b = 0$.

Proof:
Since $a^2 = b^2$, then $(a + b)(a - b) = (a - b)(a - b) = 0$. Thus $a = b$.
And $0 = a^2 + ab + b^2 = 3b^2 = 2b^2 + b^2 = b^2$, thus $0 = b = a$.

3. Let R be a commutative ring. Let S be a commutative ring without zero-divisors. Let $\varphi: R \to S$ be a ring homomorphism with $K = \ker \varphi$. Prove that for any $a, b \in R$, if $ab \in K$, then $a \in K$ or $b \in K$.

Proof:
Let $a, b \in R$, such that $ab \in K$, then $\varphi(ab) = \varphi(a)\varphi(b) = 0_S$. Since S has no zero-divisors, then $\varphi(a) = 0_S$ or $\varphi(b) = 0_S$. Thus, $a \in K$ or $b \in K$.

4. Let $A = 2\mathbb{Z}$ and $B = 8\mathbb{Z}$.

(a) Show that the group A/B is isomorphic to the group $\mathbb{Z}_4$.

Proof:
First, note that $\forall\ a \in A$, $a = 2h$ for some $h \in \mathbb{Z}$.
Define $\varphi: A \to \mathbb{Z}_4$ by $\varphi(a) = [h]$ (where $h \in \mathbb{Z}$ and $a = 2h$).
We first will show φ is a homomorphism.
Let $a, b \in A$. Then $a = 2j, b = 2k$ for some $j, k \in \mathbb{Z}$.
$\varphi(a + b) = [j + k] = [j] + [k] = \varphi(a) = \varphi(b)$.
$\therefore$ φ is a group homomorphism.
Next, we show $B = \ker \varphi$.
Let $b \in B$, then $b = 8h$ for some $h \in \mathbb{Z}$. $\varphi(b) = [4h] = [0]$. So $b \in \ker \varphi$.
Let $k \in \ker \varphi$, then $k = 2h$ for some $h \in \mathbb{Z}$ and $\varphi(k) = [h] = [0]$. Thus, $4 \mid h$, hence $8 \mid k$.
This gives us that $k \in B$. $\therefore$ $B = \ker \varphi$.
So, by the first isomorphism theorem, A/B is isomorphic to the group $\mathbb{Z}_4$.

(b) Show that the ring A/B is not isomorphic to the ring $\mathbb{Z}_4$.

Proof:
We know $2\mathbb{Z}$ does not have a unity. Since $2\mathbb{Z}/8\mathbb{Z} = \{2i + \{8Z\}: 0 \leq i < 4\}$, and $\forall\ j \neq 0$, we have that $(2j + \{8Z\}) \bullet (2i + \{8Z\}) \neq 2i + \{8Z\}$, then $2\mathbb{Z}/8\mathbb{Z}$ does not have a unity. However, $\mathbb{Z}_4$ does have a unity, [1], hence, by basic ring homomorphism properties, A/B is not ring homomorphic to the ring $\mathbb{Z}_4$. Thus, A/B is not isomorphic to the ring $\mathbb{Z}_4$.
