Content:

1.1 Number Theory

Definition d is the *gcd* of a and b if
1) $d \in Z^+$
2) $d|a$ and $d|b$
3) If $c|a$ and $c|b$, then $c|d$.

(This says that $d = k\,c$. We favor this definition for its usefulness in rings.)

Read 1.2 to mid p. 120.
Know 1.3 well by end of first week.

Read 2.1 on your own.

2.2 Permutations

Definition A permutation on a set X is a bijection $\alpha: X \to X$.

S_X = the set of all permutations on X.
S_n = the set of all permutations on $\{1, 2, \ldots, n\}$.

Example $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. As a cycle, $\alpha = (2\ 3)$.

Example $\alpha = (1\ 3\ 4\ 2)$, $\beta = (213)$, then $\alpha\beta(1) = \alpha(3) = 4$ and $\beta\alpha(1) = \beta(3) = 2$. So, order matters.

Example $(1)(142) = (142)(1)$ So, order doesn't matter if α is the identity.
$(12)(354) = (354)(12)$ So, order doesn't matter if cycles are disjoint. That is, disjoint cycles commute.

Example $\alpha = (12)(354) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$

Example Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$. Then $\beta = (132)(45)$. And $\beta = (132)(45)(6)$ in S_6.

Note: We can always write any permutation as a product of disjoint cycles. Moreover, this decomposition is unique with the exception of order (like a prime factorization of an integer).

Example The inverse of (132) is (231) as (132) (231) = (1).

Definition For $\alpha, \beta \in S_n$, α and β have the same cycle structure if their complete factorizations have the same number of *r*-cycles for each *r*.

Example (13)(425) and (23)(154) have the same cycle structure in S_5.
(2)(1345) has a different cycle structure than the above example.

Question How many 4-cycles are there in S_6?

$$\frac{\underline{6} \bullet \underline{5} \bullet \underline{4} \bullet \underline{3}}{4}$$

We divide by 4 because there are 4 orderings for each combination.

Question How many permutations are there with cycle structure (12)(34)(5)?

$$\frac{\frac{(\underline{5} \bullet \underline{4})}{2} \bullet \frac{(\underline{3} \bullet \underline{2})}{2} \bullet (\underline{1})}{2} = 15 = \frac{5 \bullet 4 \bullet 3 \bullet 2}{2 \bullet 2 \bullet 2}$$

Question List all possible cycle structures in S_5.

(12)(34)(5)	15
(1)(2)(3)(4)(5)	1
(12345)	24
(12)(345)	20
(12)(3)(4)(5)	10
(1234)(5)	30
(123)(4)(5)	<u>20</u>
	120 = 5 • 4 • 3 • 2 • 1

Example $\alpha = (124), \gamma = (13)$, then $\alpha\gamma\alpha^{-1} = (124)(13)(142) = (23) = \beta$
(β, γ are conjugates)

(1) Note that γ and $\alpha\gamma\alpha^{-1}$ have the same cycle structure.
(2) Note that $\alpha\gamma\alpha^{-1} = (\alpha(1)\ \ \alpha(3))$.

Definition $\beta,\gamma \in S_n$. We say β is a conjugate of γ if $\beta = \alpha\gamma\alpha^{-1}$ for some $\alpha \in S_n$.

Brief discussion of (1): $\beta = \alpha\gamma\alpha^{-1} \Leftrightarrow \beta\alpha = \alpha\gamma \Leftrightarrow \beta\alpha(i) = \alpha\gamma(i)$ for all i.
Suppose γ: $i \mapsto j$. Then β: $\alpha(i) \mapsto \alpha(j)$.
(Any time γ moves something, β moves something too. So we have the same cycle structure.)

Homework 2.2 #2, 7, 9, 11, **15**, +**Theorem Proof** (thm will be given Thursday).
Due Mon 9/14/09