Content:

Theorem	Let β and γ be permutations in S_n. Then β and γ are conjugates $\Leftrightarrow$ they have the same cycle structure.
Theorem	Any permutation can be expressed as a product of transpositions.
Definition	$\text{sgn}(\alpha)$
Definition	even/odd permutation
Lemma	Let $\alpha \in S_n$. Let $\tau \in S_n$ be a transposition. Then $\text{sgn}(\tau\alpha) = -\text{sgn}(\alpha)$.
Theorem	$\text{sgn}(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$, $\forall\, \alpha, \beta \in S_n$.
Theorem	Let $\alpha \in S_n$. (1) α is even iff $\text{sgn}(\alpha) = 1$. (2) α is odd iff $\text{sgn}(\alpha) = -1$.
Corollary	If α, β have the same parity, then $\alpha\beta$ is even. If α, β have different parities, then $\alpha\beta$ is odd.

Warm-up $\alpha = (132)$, $\gamma = (15)(243)$, find $\alpha\gamma\alpha^{-1}$.
As shown Tuesday, $\alpha\gamma\alpha^{-1} = (\alpha(1)\ \ \alpha(5))(\alpha(2)\ \ \alpha(4)\ \ \alpha(3)) = (35)(142)$.

Theorem Let β and γ be permutations in S_n. Then β and γ are conjugates $\Leftrightarrow$ they have the same cycle structure.

Proof (sketch):
$\Rightarrow$: Suppose $\exists\ \alpha$ such that $\beta = \alpha\gamma\alpha^{-1}$, then $\beta\alpha = \alpha\gamma$ and $\beta\alpha(i) = \alpha\gamma(i)$.
If $\gamma: i \mapsto j$, then $\beta: \alpha(i) \mapsto \alpha(j)$. (Expand this to the general case).

General case:
If γ fixes i, then $\beta\alpha(i) = \alpha\gamma(i) = \alpha(i)$. $\therefore$ β fixes i.
Let $\gamma = \sigma_1\, \sigma_2 \cdots (i_1\, i_2 \ldots i_r) \cdots \sigma_t$ be the complete factorization of γ into disjoint cycles.
Then $\gamma(i_1) = i_2$ and $\beta\alpha(i_1) = \alpha\gamma(i_1) = \alpha(i_2)$. $\therefore$ $\beta: \alpha(i_1) \mapsto \alpha(i_2)$.
$\therefore$ One of the cycles of β is $(\alpha(i_1)\ \alpha(i_2) \ldots \alpha(i_r))$.
Thus, we get the same cycle structure and a quick computation method.

Example for $\Leftarrow$: $\gamma = (c_1\, c_2\, c_3)(c_4\, c_5)$
$\beta = (d_1\, d_2\, d_3)(d_4\, d_5)$
Define α as follows: $\alpha(c_1) = d_1$, $\alpha(c_2) = d_2$, …, then
$\beta = (\alpha(c_1)\ \ \alpha(c_2)\ \alpha(c_3))(\alpha(c_4)\ \alpha(c_5))$
If we have the same structure, we can pair each cycle in β with its partner in γ. Construct α from this pairing.

Another useful factorization in S_n is into transpositions.

Example $(1432) = (12)(13)(14)$ or $(14)(43)(32)$.
General case: $(i_1\, i_2 \ldots i_r) = (i_1\, i_r) \ldots (i_1\, i_3)(i_1\, i_2)$ or $(i_1\, i_2)(i_2\, i_3) \ldots (i_{r-1}\, i_r)$.

Theorem Any permutation can be expressed as a product of transpositions.

Note This decomposition is neither unique nor commutative.

Two types of factorizations, complete factorizations and products of transpositions, give rise to two definitions, signum and even/odd, respectively.

Definition (1) Let $\alpha \in S_n$ and $\alpha = \beta_1 \cdots \beta_t$ be a complete factorization into disjoint cycles, then $sgn(\alpha) = (-1)^{n-t}$.

Example $\alpha = (12)(3)(4)(5) \in S_5$. $\text{sgn}(\alpha) = (-1)^{5-4} = -1$.

Note $\text{sgn}(\alpha)$ is well-defined as factorizations are unique.

Definition (2) A permutation in S_n is even if it can be expressed as a product of an even amount of transpositions, otherwise it is odd.

Example (1) $(143) = (13)(14)$ So (143) is even.
Example (2) $(13)(243) = (13)(24)(43)$
Does this mean it is odd? We don't know from the definition.
We would have to show there does not exist an even decomposition.

Example $\alpha = (12)(13524) = (135)(24) \in S_5$.
So $\text{sgn}(\alpha) = (-1)^{5-2} = (-1)^3$.

$\text{sgn}(13524) = (-1)^{5-1} = (-1)^4$.

The difference between these two is a single transposition.

Lemma Let $\alpha \in S_n$. Let $\tau \in S_n$ be a transposition. Then $\text{sgn}(\tau\alpha) = -\text{sgn}(\alpha)$.

Proof:
$(a\ b)(a\ c_1 \dots c_k\ b\ d_1 \dots d_L) = (a\ c_1 \dots c_k)(b\ d_1 \dots d_L)$
So $(a\ b)(a\ b)(a\ c_1 \dots c_k\ b\ d_1 \dots d_L) = (a\ b)(a\ c_1 \dots c_k)(b\ d_1 \dots d_L)$
$\therefore\ (a\ c_1 \dots c_k\ b\ d_1 \dots d_L) = (a\ b)(a\ c_1 \dots c_k)(b\ d_1 \dots d_L)$

Let $\alpha = \beta_1 \beta_2 \cdots \beta_t$ be a complete factorization into disjoint cycles.
Then $(a\ b)\ \alpha = (a\ b)\ \beta_1 \beta_2 \cdots \beta_t$.
Note: This is not a complete factorization.
We'll get cases depending on what α does to a and b.
(i) Example (34)(12)(3)(4)(5) Get rid of (3)(4). Finish this on your own.
(ii) The case we'll do is $\alpha(a) \neq a$, $\alpha(b) \neq b$ and a, b live in the same cycle.
$\alpha = \beta_1 \beta_2 \cdots \beta_t$. Without loss of generality, $\beta_1 = (a\ c_1 \dots c_k\ b\ d_1 \dots d_L)$
Then
$$\begin{aligned}(a\ b)\ \alpha &= (a\ b)\ \beta_1 \beta_2 \cdots \beta_t \\ &= (a\ b)(a\ c_1 \dots c_k\ b\ d_1 \dots d_L)\beta_2 \cdots \beta_t \\ &= (a\ c_1 \dots c_k)(b\ d_1 \dots d_L)\beta_2 \cdots \beta_t\end{aligned}$$
$\therefore\ \text{sgn}(\alpha) = (-1)^{n-t} = -(-1)^{n-(t+1)} = -\text{sgn}((ab)\alpha) = -\text{sgn}(\tau\alpha)$.

Theorem $\text{sgn}(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$, $\forall\ \alpha, \beta \in S_n$.

Proof: Hand in as homework. Use lemma above.

Theorem Let $\alpha \in S_n$.
(1) α is even iff $\text{sgn}(\alpha) = 1$.
(2) α is odd iff $\text{sgn}(\alpha) = -1$.

(1) Proof:
$\Rightarrow$: Assume α is even. $\alpha = \tau_1 \cdots \tau_k$ is a product of transpositions and k is even.
$\text{sgn}(\alpha) = \text{sgn}(\tau_1)\ \text{sgn}(\tau_2) \cdots \text{sgn}(\tau_k) = (-1)(-1)\cdots(-1) = (-1)^k = 1$.

$\Leftarrow$: Do on your own.

Assume $\text{sgn}(\alpha) = 1$. Since we can write $\alpha = \tau_1 \cdots \tau_k$, a product of transpositions, and $\text{sgn}(\alpha) = \text{sgn}(\tau_1)\ \text{sgn}(\tau_2) \cdots \text{sgn}(\tau_k) = (-1)(-1)\cdots(-1) = (-1)^k = 1$, then k must be even.

(2) Make $\alpha = \tau_1 \cdots \tau_k$ arbitrary. This shows that k must always be even or always be odd.

$\Rightarrow$: Assume α is odd. Then α is not even, and so $\text{sgn}(\alpha) \neq 1$; that is, $\text{sgn}(\alpha) = -1$.
$\Leftarrow$: Assume $\text{sgn}(\alpha) = -1$. Since we can write $\alpha = \tau_1 \cdots \tau_k$, a product of transpositions, and $\text{sgn}(\alpha) = \text{sgn}(\tau_1)\ \text{sgn}(\tau_2) \cdots \text{sgn}(\tau_k) = (-1)(-1)\cdots(-1) = (-1)^k = -1$, then k must be odd.

Moral Odd really does mean an odd number of transpositions.

Odd+odd = even
Even+even = even
Even+odd = odd

Corollary If α, β have the same parity, then $\alpha\beta$ is even. If α, β have different parities, then $\alpha\beta$ is odd.

Proof:
Do on your own.

Assume α, β have the same parity, then $\text{sgn}(\alpha) = \text{sgn}(\beta)$.
Since $\text{sgn}(\alpha\beta) = \text{sgn}(\alpha)\text{sgn}(\beta)$, then $\text{sgn}(\alpha\beta) = 1$, hence $\alpha\beta$ is even.
Assume α, β have different parities, then $\text{sgn}(\alpha) = -\text{sgn}(\beta)$ and $\text{sgn}(\alpha\beta) = -1$, hence $\alpha\beta$ is odd.
