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Warm-up

Get G be a group. Let $H \triangleleft G$, $K \triangleleft G$ such that $H \cap K = \{e\}$.

Prove $hk = kh \forall h \in H, \forall k \in K$.

Proof:

Since $H \triangleleft G$, $h(kh^{-1}k^{-1}) \in H$.

Since $K \triangleleft G$, $(hkh^{-1})k^{-1} \in K$.

$\therefore hkh^{-1}k^{-1} \in H \cap K = \{e\}$.

$\therefore hkh^{-1}k^{-1} = \{e\}$.

$\therefore hk = kh$.

Proposition If G is a finite Abelian group and $d \mid |G|$, then G contains a subgroup of order d .

Proof:

First, we'll prove that if $p \mid |G|$ where p is prime, then G has a subgroup of order p .

Let $|G| = n$. We will prove by induction on n .

Base case, $n = 1$. \checkmark

Induction hypothesis: Let $s < n$, then assume any group of order s such that $p \mid s$ has a subgroup of order p .

Let $a \in G$ where $\circ(a) = k > 1$. If $p \mid k$ then $\circ(a^{k/p}) = p$ (by the gcd theorem).

Let $H = \langle a^{k/p} \rangle$. There's our subgroup. Suppose $p \nmid k$, then let $H = \langle a \rangle$.

So $|H| = k$. We know $H \triangleleft G$ since G is Abelian. So G/H is a group and $|G/H| = n/k < n$. We know $p \mid n$ and $p \nmid k$. Thus $p \mid n/k$.

By our induction hypothesis, we have a subgroup of G/H of order p . So, by the Correspondence Theorem, $\exists H' \leq G$ such that $|H'/H| = p$. Since p is prime, then H'/H is cyclic. $\therefore \exists b \in G$ such that $\langle bH \rangle = H'/H$.

Since $|H'/H| = p$, then $\circ(bH) = p$.

Since $\pi: G \rightarrow G/H$ takes b to bH we know $\circ(\pi(b)) \mid \circ(b)$.

Since $p \mid \circ(b)$, then $\circ(b^{p/p}) = p$. So $H'' = \langle b^{p/p} \rangle$, hence $|H''| = |\langle b^{p/p} \rangle| = p$.

Read the rest of the proof in the book.

So, from this, we know if $p \mid |G|$ then $\exists a \in G$ such that $\circ(a) = p$.

But, this is not true for any divisor $d \mid |G|$. Our counterexample is $\mathbb{Z}_2 \times \mathbb{Z}_3$. There is no element of order 6 even though $|\mathbb{Z}_2 \times \mathbb{Z}_3| = 6$.

Presentation of Groups Using Generators and Relations

Definition Let X be a set, then $\langle X \rangle$ is the smallest subgroup containing all elements of X .

$\langle x, y \rangle$ is the smallest subgroup containing x and y .

We know it must contain: $1, x, y, x^k, y^j, xy, xy^7, x^6y^8, xyx, \dots$

Throw in relations:

Example $\langle x, y \mid xy = yx \rangle$. In this case, any element can be expressed as $x^k y^j$.

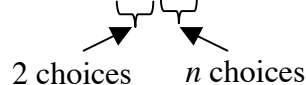
Example $D_{2n} = \langle a, b \mid b^n = 1 = a^2, aba = b^{-1} \rangle$

$$ab = b^{-1}a^{-1} = b^{n-1}a$$

So $a^k b^j a^m$ can be expressed as $a^{k-1} a b b^{j-1} a^m$.

Any element can be written as $a^k b^j$ where $k \in \{0, 1\}$ and $j \in \{0, 1, \dots, n-1\}$

In D_{2n} , elements are of the form: $a^k b^j$ which tells us how many elements.



-or a total of $2n$ choices

Direct Products

$$HK = \{hk \mid h \in H, k \in K\}.$$

Example \mathbb{Z}_6

Find $H \leq \mathbb{Z}_6$ and $K \leq \mathbb{Z}_6$ such that $H + K = \mathbb{Z}_6$.

$H = \{0, 3\} \leq \mathbb{Z}_6$ and $K = \{0, 2, 4\} \leq \mathbb{Z}_6$
 And $H + K = \mathbb{Z}_6$.

$H + K \cong H \times K$? Yes.

$H \triangleleft \mathbb{Z}_6, K \triangleleft \mathbb{Z}_6$ and $H \cap K = \{e\}$

Example S_3

Find $H \leq S_3$ and $K \leq S_3$ such that $HK = S_3$.

$H = \langle (12) \rangle \leq S_3$ and $K = \langle (123) \rangle \leq S_3$
 And $HK = S_3$.

$HK \cong H \times K$? No.

$H \cap K = \{e\}$, but $\sim (H \triangleleft S_3, K \triangleleft S_3)$

Definition $G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$
 We call this the *external direct product*.

Definition If G is a group with $H \triangleleft G$ and $K \triangleleft G$, $HK = G$, and $H \cap K = \{e\}$, then G is said to be the *internal direct product* of H and K .

From handout:

4 (a) Let $x = hk \in G$. $\phi: H \cap K \rightarrow \{(h', k') \in H \times K \mid h'k' = x\}$.

This is all ordered pairs whose product of coordinates is hk .

This gives motivation for $H \cap K = \{e\}$.

4 (c) Rewritten: $|HK| |H \cap K| = |H \times K| = |H| |K|$.
 $\underbrace{\quad}_x$ -how many ways we can write x

Theorem If G is an internal direct product of H and K , then $G \cong H \times K$.

Theorem If $G \cong G_1 \times G_2$, \exists normal subgroups H_1 and H_2 of G such that G is the internal direct product of H_1 and H_2 .