Content:

Proposition	If G is a finite Abelian group and $d \mid \lvert G \rvert$, then G contains a subgroup of order d.
Definition	$\langle X \rangle$ Smallest subgroup of X containing all elements of X.
Definition	External Direct Product
Definition	Internal Direct Product
Theorem	If G is an internal direct product of H and K, then $G \cong H \times K$.
Theorem	If $G \cong G_1 \times G_2$, $\exists$ normal subgroups H_1 and H_2 of G such that G is the internal direct product of H_1 and H_2.

Warm-up

Get G be a group. Let $H \lhd G, K \lhd G$ such that $H \cap K = \{e\}$.
Prove $hk = kh \ \forall \ h \in H, \forall \ k \in K$.

Proof:
Since $H \lhd G$, $h(kh^{-1}k^{-1}) \in H$.
Since $K \lhd G$, $(hkh^{-1})k^{-1} \in K$.
$\therefore hkh^{-1}k^{-1} \in H \cap K = \{e\}$.
$\therefore hkh^{-1}k^{-1} = \{e\}$.
$\therefore hk = kh$.

Proposition If G is a finite Abelian group and $d \mid \lvert G \rvert$, then G contains a subgroup of order d.

Proof:
First, we'll prove that if $p \mid \lvert G \rvert$ where p is prime, then G has a subgroup of order p.
Let $\lvert G \rvert = n$. We will prove by induction on n.
Base case, $n = 1$. √
Induction hypothesis: Let $s < n$, then assume any group of order s such that $p \mid s$ has a subgroup of order p.

Let $a \in G$ where $\circ(a) = k > 1$. If $p \mid k$ then $\circ(a^{\frac{k}{p}}) = p$ (by the gcd theorem).
Let $H = \langle a^{\frac{k}{p}} \rangle$. There's our subgroup. Suppose $p \nmid k$, then let $H = \langle a \rangle$.
So $\lvert H \rvert = k$. We know $H \lhd G$ since G is Abelian. So G/H is a group and $\lvert G/H \rvert = n/k < n$. We know $p \mid n$ and $p \nmid k$. Thus $p \mid n/k$.
By our induction hypothesis, we have a subgroup of G/H of order p. So, by the Correspondence Theorem, $\exists \ H' \leq G$ such that $\lvert H'/H \rvert = p$. Since p is prime, then H'/H is cyclic. $\therefore \ \exists \ b \in G$ such that $\langle bH \rangle = H'/H$.
Since $\lvert H'/H \rvert = p$, then $\circ(bH) = p$.
Since $\pi{:}G \to G/H$ takes b to bH we know $\circ(\pi(b)) \mid \circ(b)$.
Since $p \mid \circ(b)$, then $\circ(b^{\frac{\circ(b)}{p}}) = p$. So $H'' = \langle b^{\frac{\circ(b)}{p}} \rangle$, hence $\lvert H'' \rvert = \lvert \langle b^{\frac{\circ(b)}{p}} \rangle \rvert = p$.
Read the rest of the proof in the book.

So, from this, we know if $p \mid |G|$ then $\exists\ a \in G$ such that $\circ(a) = p$.
But, this is not true for any divisor $d \mid |G|$. Our counterexample is $\mathbb{Z}_2 \times \mathbb{Z}_3$. There is no element of order 6 even though $|\mathbb{Z}_2 \times \mathbb{Z}_3| = 6$.

Presentation of Groups Using Generators and Relations

Definition Let X be a set, then $\langle X \rangle$ is the smallest subgroup containing all elements of X.

$\langle x, y \rangle$ is the smallest subgroup containing x and y.
We know it must contain: $1, x, y, x^k, y^j, xy, xy^7, x^6y^8, xyx, \ldots$

Throw in relations:
Example $\langle x, y \mid xy = yx \rangle$. In this case, any element can be expressed as $x^k y^j$.

Example $D_{2n} = \langle a, b \mid b^n = 1 = a^2, aba = b^{-1} \rangle$

$$\begin{aligned} ab &= b^{-1}a^{-1} \\ &= b^{n-1}a \end{aligned}$$

So $a^k b^j a^m$ can be expressed as $a^{k-1}abb^{j-1}a^m$.
Any element can be written as $a^k b^j$ where $k \in \{0, 1\}$ and $j \in \{0, 1, \ldots, n-1\}$

In D_{2n}, elements are of the form: $a^k\, b^j$ which tells us how many elements.

(a^k: 2 choices; b^j: n choices)

-or a total of $2n$ choices

Direct Products

$$HK = \{hk \mid h \in H, k \in K\}.$$

Example $\mathbb{Z}_6$
Find $H \leq \mathbb{Z}_6$ and $K \leq \mathbb{Z}_6$ such that $H + K = \mathbb{Z}_6$.

$H = \{0, 3\} \leq \mathbb{Z}_6$ and $K = \{0, 2, 4\} \leq \mathbb{Z}_6$
And $H + K = \mathbb{Z}_6$.

$H + K \cong H \times K$? Yes.
$H \lhd \mathbb{Z}_6$, $K \lhd \mathbb{Z}_6$ and $H \cap K = \{e\}$

Example S_3
Find $H \leq S_3$ and $K \leq S_3$ such that $HK = S_3$.

$H = \langle (12) \rangle \leq S_3$ and $K = \langle (123) \rangle \leq S_3$
And $HK = S_3$.

$HK \cong H \times K$? No.
$H \cap K = \{e\}$, but $\sim(H \lhd S_3, K \lhd S_3)$

Definition $G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$
We call this the *external direct product*.

Definition If G is a group with $H \lhd G$ and $K \lhd G$, $HK = G$, and $H \cap K = \{e\}$, then G is said to be the *internal direct product* of H and K.

From handout:

4 (a) Let $x = hk \in G$. $\phi: H \cap K \to \{(h', k') \in H \times K \mid h'k' = x\}$.

This is all ordered pairs whose product of coordinates is hk.

This gives motivation for $H \cap K = \{e\}$.

4 (c) Rewritten: $|HK|\ |H \cap K| = |H \times K| = |H|\,|K|$.

x -how many ways we can write x

Theorem If G is an internal direct product of H and K, then $G \cong H \times K$.

Theorem If $G \cong G_1 \times G_2$, $\exists$ normal subgroups H_1 and H_2 of G such that G is the internal direct product of H_1 and H_2.