Content:

Proposition G has a unique Sylow p-subgroup $\Leftrightarrow P \lhd G$.

Theorem If all Sylow p-subgroups are normal to G, then G is isomorphic to a direct product of its Sylow p-subproups.

Proposition Let G be a nonabelian group of order $|G| = p^e m$, $(p, m) = 1$, p a prime. If $p^e \nmid (m-1)!$, then G is not simple.

Theorem All groups of order 30 are not simple.

Recall $|G| = p^k n$

- All Sylow p-subgroups are conjugate.
- n_p = the number of Sylow p-subgroups = $[G:N_G(P)]$
- $n_p \equiv 1 \pmod p$
- $n_p \mid n$.

Proposition G has a unique Sylow p-subgroup $\Leftrightarrow P \lhd G$.

Proof:

Let P be a Sylow p-subgroup of G.

$P \lhd G \Leftrightarrow gPg^{-1} = P \; \forall \, g \in G$

$\Leftrightarrow \mathcal{O}(P) = \{P\}$ where G acts by conjugation on its subgroups

$\Leftrightarrow |\mathcal{O}(P)| = 1$

$\Leftrightarrow n_p = 1$

$\Leftrightarrow P$ is unique.

So if we find $n_p = 1$, then the group is not simple. It has a normal subgroup.

Example Prove any group, G, of order 40 is not simple.

Proof:

$40 = 2^2 \bullet 5$

So $\underbrace{n_5 \equiv 1 \pmod 5}_{n_5 = 1, 6, 11, \ldots}$ and $\underbrace{n_5 \mid 8}_{n_5 = 1, 2, 4, 8}$.

$\therefore$ $n_5 = 1$. So $P_5 \lhd G$, Hence G is not simple.

Theorem If all Sylow p-subgroups are normal to G, then G is isomorphic to a direct product of its Sylow p-subproups.

Proof:

See proof of Theorem 6 in Finite Abelian Groups handout (Let G be a finite Abelian group such that $|G| = p_1^{k_1} \cdots p_l^{k_l}$ where each of the p_i are distinct primes.

Then $G \cong G(p_1) \times \cdots \times G(p_l)$.). The proof of Theorem 6 shows that G is the internal direct product of it Sylow p_i subgroups and therefore by Theorem 3, $G \cong G(p_1) \times \cdots \times G(p_l)$. Since we don't have G is Abelian in this case, we need the Sylow p-subgroups to be normal. Then the proof is the same as for Theorem 6.

Example Classify all groups of order 35.

Proof:

Let $|G| = 35$. $35 = 5 \cdot 7$.

So $\underbrace{n_5 \mid 7}_{n_5 = 1,\,7}$ and $\underbrace{n_5 \equiv 1 \pmod 5}_{n_5 = 1,\,6,\,11,\,\ldots}$. Also $\underbrace{n_7 \mid 5}_{n_7 = 1,\,5}$ and $\underbrace{n_7 \equiv 1 \pmod 7}_{n_7 = 1,\,8,\,15,\,\ldots}$

$\therefore$ $P_5 \lhd G$ and $P_7 \lhd G$.

By the previous theorem, $G \cong P_5 \times P_7 \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35}$.

Example Classify all groups of order 10.

Proof:

Let $|G| = 10$. $10 = 2 \cdot 5$

So $\underbrace{n_2 \mid 5}_{n_2 = 1,\,5}$ and $\underbrace{n_5 \equiv 1 \pmod 2}_{n_2 = 1,\,3,\,5,\,\ldots}$. Also $\underbrace{n_5 \mid 2}_{n_5 = 1,\,2}$ and $\underbrace{n_5 \equiv 1 \pmod 5}_{n_5 = 1,\,6,\,11,\,\ldots}$

So $n_2 = 1$ or 5 and $n_5 = 1$.

If $n_2 = 1$, then $P_2 \lhd G$. Since $n_5 = 1$, then $P_5 \lhd G$.

So $G \cong P_2 \times P_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10}$.

If $n_2 = 5$, then since $n_5 = 1$, we have $|P_5| = 5$,

hence for some $a \in G$, $P_5 = \langle a \rangle$.

And since we have 5 subgroups of order 2,

then by Cauchy's theorem, $\exists\, b \in G$ such that $\circ(b) = 2$.

Since $P_5 \lhd G$, then $bab^{-1} \in P_5$. So $bab^{-1} = a^k$ for some $k \in \{0, 1, 2, 3, 4\}$.

Since $\circ(b) = 2$, then $b = b^{-1}$, hence $bab^{-1} = bab = a^k$.

So $a = ba^kb^{-1} = \underbrace{(bab^{-1})(bab^{-1}) \cdots (bab^{-1})}_{k \text{ factors}} = (bab^{-1})^k = (bab)^k = (a^k)^k = a^{k^2}$.

Thus, $a^{k^2-1} = e$. Since $\circ(a) = 5$, then $5 \mid k^2 - 1$.

Since $k \in \{0, 1, 2, 3, 4\}$, we get $k = 1$ or 4.

If $k = 1$, then $a = bab$, hence $ab = ba$.

It can be easily checked that $a^2b = ba^2$, $a^3b = ba^3$, and $a^4b = ba^4$.

$\therefore$ $\forall\, g \in G$, $gbg^{-1} \in P_2$.

$\therefore$ $P_2 \lhd G$. Hence we get $G \cong \mathbb{Z}_{10}$.

If $k = 4$, then $bab = a^4 = a^{-1}$ and $a^5 = e = b^2$.

And this is the presentation definition of D_{10}.

So there is a subgroup of G that is isomorphic to D_{10}.

And since $|D_{10}| = |G|$, then $G \cong D_{10}$.

Thus, if $|G| = 10$, then $G \cong \mathbb{Z}_{10}$ or $G \cong D_{10}$.

Recall — From homework #98, If $|G| = pm$ where p is prime and $p > m$, then G is not simple.

Proposition — Let G be a nonabelian group of order $|G| = p^e m$, $(p, m) = 1$, p a prime. If $p^e \nmid (m-1)!$, then G is not simple.

Proof:

Let P be a Sylow p-subgroup of G. Assume $p^e \nmid (m-1)!$. So $[G{:}P] = m$.
Since $p^e \nmid (m-1)!$ then $|G| = p^e m \nmid m(m-1)! = m!$
$\therefore$ By the Index Factorial theorem, G is not simple.

This proposition shows that group of order 12, 18, 24, 36, 45, 48, 50, 54 are not simple.

Example — Prove all groups of order 54 are not simple.

Proof:

Let $|G| = 54$. $54 = 2 \bullet 3^3$. $3^3 \mid (2-1)! = 1$. $\therefore$ G is not simple.

The only groups left of order less than 60 are 30 and 56.
The proof is similar for both. Let's prove for 30.

Theorem — All groups of order 30 are not simple.

Proof:

Let $|G| = 30$. $30 = 2 \bullet 3 \bullet 5$.

$$\underbrace{n_2 \mid 15}_{n_2 = 1, 3, 5, 15} \text{ and } \underbrace{n_2 \equiv 1 \pmod 2}_{n_2 = 1, 3, 5, \ldots};\ \underbrace{n_3 \mid 10}_{n_3 = 1, 2, 5, 10} \text{ and } \underbrace{n_3 \equiv 1 \pmod 3}_{n_3 = 1, 4, 7, 11};\ \underbrace{n_5 \mid 6}_{n_5 = 1, 2, 3, 6} \text{ and } \underbrace{n_5 \equiv 1 \pmod 5}_{n_5 = 1, 6, 11, 16, \ldots}.$$

$\therefore$ $n_2 = 1, 3, 5$, or 15. $n_3 = 1$ or 10. $n_5 = 1$ or 6.

Suppose $n_2 \neq 1$, $n_3 \neq 1$, and $n_5 \neq 1$.

Then $n_5 = 6$, which means G has 6 subgroups of order 5.

This gives us 24 distinct non-identity elements of order 5.

Also $n_3 = 10$, which means G has 10 subgroups of order 2, each of which has a distinct element of order 2.

Thus, we have run out of elements as $|G| = 30$.

$\therefore$ At least one of n_2, n_3, or n_5 must be 1. Hence G is not simple.
