Content:

Example $2\mathbb{Z} \subseteq$ Z (i.e. 2 $\mathbb{Z}$ is a subring of $\mathbb{Z}$). Even though 2 $\mathbb{Z}$ does not have a unity, it is still a subring (contrary to Rotman's definition)

Example $S = \{0, 2, 4, 6, 8\} \subseteq \mathbb{Z}_{10}$. In $\mathbb{Z}_{10}$, 1 is the unity, but in S, 6 is the unity.

Definition *R*[*x*] is the set of all polynomials over the ring, R.

Example $(2/3)x^2 + x - 5 \in \mathbb{Q}[x]$.

3.3 Polynomials

For this section, we will assume R is a commutative ring with unity.

Definition The *Power Series Ring*, $R[[x]]$, is the set of all infinite sequences with entries in R. $R[[x]] = \{(a_0, a_1, a_2, \ldots) : a_i \in R\}$.

Our goal is to prove $R[[x]]$ is a ring.

Definition *Addition* in $R[[x]]$ is defined as
$(a_0, a_1, a_2, \ldots) + (b_0, b_1, b_2, \ldots) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$
and can be denoted $(a_i) + (b_i) = (a_i + b_i)$.

Definition *Multiplication* in $R[[x]]$ is defined as

$(a_i)(b_i) = (c_i)$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$ or $c_i = \sum_{j+k=i} a_j b_k$.

Example Let $(a_i) = (1, 2, 4, 0, 0, \ldots)$ and $(b_i) = (3, 4, 0, 0, \ldots)$

Then $c_0 = \sum_{j+k=0} a_j b_k = a_0 b_0 = 1 \bullet 3 = 3$.

$c_1 = \sum_{j+k=1} a_j b_k = a_0 b_1 + a_1 b_0 = 1 \bullet 4 + 2 \bullet 3 = 10$.

$c_2 = \sum_{j+k=2} a_j b_k = a_0 b_2 + a_1 b_1 + a_2 b_0 + = 1 \bullet 0 + 2 \bullet 4 + 4 \bullet 3 = 23$.

$c_3 = \sum_{j+k=3} a_j b_k = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 1 \bullet 0 + 2 \bullet 0 + 4 \bullet 4 + 0 \bullet 3 = 20$.

$c_4 = \sum_{j+k=4} a_j b_k = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 = 1 \bullet 0 + 2 \bullet 0 + 4 \bullet 0 + 0 \bullet 4 + 0 \bullet 3 = 0$.

This tells us that $(1 + 2x + 5x^2)(3 + 4x) = 3 + 10x + 24x^2 + 20x^3$.

Example Prove $c_k = 0$ if $k \ge 4$ (in the above example).

Proof:
Note that $a_r = 0$ if $r \ge 3$ and $b_s = 0$ if $s \ge 2$.
$c_k = \sum_{h+j=k} a_j b_k$. Suppose $h \ge 3$. Then $a_h b_j = 0 \bullet b_j = 0$.
Suppose $h < 3$. Then $a_h b_j = a_h \bullet 0 = 0$.
(Since $k \ge 4 \Rightarrow h + j \ge 4$ and $h < 3 \Rightarrow 3 + j > h + j \ge 4$, hence $j \ge 2$.)
Thus, $c_k = \sum_{h+j=k} a_j b_k = 0$ if $k \ge 4$.

Theorem $R[[x]]$ is a commutative ring with unity.

Proof:
Since addition is component-wise, we get that $R[[x]]$ is an Abelian group from the fact that R is an Abelian group under $+$.
Need to show: $(a_i)[(b_i)(c_i)] = [(a_i)(b_i)](c_i)$.

$$(a_i)[(b_i)(c_i)] = (a_i)[(d_i)] \text{ where } d_i = \sum_{j+k=i} b_j c_k$$
$$= (e_i) \text{ where } e_i = \sum_{r+s=i} a_r d_s$$
$$= \sum_{r+s=i} a_r \sum_{j+k=s} b_j c_k = \sum_{r+s=i} \sum_{j+k=s} a_r b_j c_k = \sum_{r+j+k=s} a_r b_j c_k$$
$$[(a_i)(b_i)](c_i) = (f_i)(c_i) \text{ where } f_i = \sum_{j+k=i} a_j b_k$$
$$= (g_i) \text{ where } g_i = \sum_{r+s=i} f_r c_s$$
$$= \sum_{r+s=i} \left[\sum_{j+k=r} a_j b_k \right] c_s = \sum_{r+s=i} \sum_{j+k=s} a_r b_j c_k = \sum_{r+j+k=s} a_r b_j c_k$$

After relabeling, we see the 2 sums are equal.
Multiplicative identity (unity) in $R[[x]]$: $(1, 0, 0, \ldots)$.
(Since $(1,0,0,\ldots)(a_i) = (1 \bullet a_0, 1 \bullet a_1 + 0 \bullet a_0, 1 \bullet a_2 + 0 \bullet a_1 + 0 \bullet a_0, \ldots) = (a_i)$.)
Commutativity is easy because a_j, b_k live in a commutative ring.
Distributive Property: You do it.
$(a_i)[(b_i) + (c_i)] = (a_i)(b_i + c_i) = (d_i)$ where $d_i =$
$$\sum_{j+k=i} a_j (b_k + c_k) = \sum_{j+k=i} a_j b_k + a_j c_k = \sum_{j+k=i} a_j b_k + \sum_{j+k=i} a_j c_k$$
And $(a_i)(b_i) + (a_i)(c_i) = (f_i) + (g_i) = (f_i + g_i)$ where
$f_i = \sum_{j+k=i} a_j b_k$ and $g_i = \sum_{j+k=i} a_j c_k$.
Since $d_i = f_i + g_i$, then $(a_i)[(b_i) + (c_i)] = (a_i)(b_i) + (a_i)(c_i)$.
Additive identity: $(0, 0, \ldots)$
Additive inverse of (a_i) is $(-a_i)$.

$R[[x]]$ is a not a field.

Another special element in $R[[x]]$ is $x = (0, 1, 0, 0, \ldots)$.
And $x^2 = (0, 0, 1, 0, 0, \ldots)$ since $(0, 1, 0, 0, \ldots) \bullet (0, 1, 0, 0, 0, \ldots) =$
$(0 \bullet 0, 0 \bullet 1 + 1 \bullet 0, 0 \bullet 0 + 1 \bullet 1 + 0 \bullet 0, \ldots)$.

Example $(2, 1, 0, 3, 4, 0, 0, \ldots)$
$(2, 0, \ldots) + (0, 1, 0, \ldots) + (0, 0, 0, 3, 0, \ldots) + (0, 0, 0, 0, 4, 0, \ldots)$
And $(0, 0, 0, 3, 0, \ldots)$
$= (0, 0, 0, 1, 0, \ldots) + (0, 0, 0, 1, 0, \ldots) + (0, 0, 0, 1, 0, \ldots)$
$= 3x^3$.
By repeating this procedure on $(0, 0, 0, 0, 4, 0, \ldots)$, we have that
$(2, 1, 0, 3, 4, 0, 0, \ldots) = 2 + x + 0x^2 + 3x^3 + 4x^4$.

Definition The *polynomial ring over R* is
$R[x] = \{(a_i) \in R[[x]] \mid (\exists\, M \in \mathbb{N})(\forall i \geq M)(a_i = 0)\}$.

Note that $(1, 2, 3, 4, 5, \ldots) \in R[[x]]$, but $(1, 2, 3, 4, 5, \ldots) \notin R[x]$.