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Definition Let $(s_i) \in R[x]$. Then $(s_i) = (s_0, s_1, \ldots, s_n, 0, 0, \ldots)$.
If $s_n \neq 0$, we say the *degree* of (s_i) is n.

Note The degree of the 0 polynomial is undefined. (Some authors define it negative infinity or any negative number—We will consider it undefined.)

Question $\deg(f + g) = ?$
Consider $(-x^2 + 1) + x^2 = 1$.
$\deg(f + g) \leq \max(\deg g + \deg f)$ or is undefined if 0 polynomial.

Question $\deg(fg) = ?$
Consider $(2x)(2x) = 0$ in $\mathbb{Z}_4[x]$.
$\deg(fg) \leq \deg g + \deg f$ or is undefined.

Theorem If R is an integral domain, then $R[x]$ is an integral domain and $\deg(fg) \leq \deg g + \deg f$ if $f, g \neq 0$.

Proof: First show $R[x]$ is an integral domain. You do it.
We have proven (lecture notes 11/16/09) that $R[[x]]$ is a commutative ring with unity, so we only need to show $R[x]$ satisfies the axioms of a domain.
Let $f, g \in R[x]$ such that $fg = 0$.
Then $(f_i)(g_i) = (c_i)$ where $c_i = \Sigma_{j+k=i} f_j g_k$ and $\forall\ i \geq 0, c_i = 0$.
Suppose $f \neq 0$ and $g \neq 0$. Let $j = \min\ \{i: f_i \neq 0\}, k = \{i: g_k \neq 0\}$.
Then, (1) if $k > j$ and $j > 0$, $c_{k+j-1} = f_j g_k \neq 0$. That is,
$(0, \ldots, f_j, f_{j+1}, \ldots f_{k+j-1})(0, \ldots, g_k, g_{k+1}, \ldots g_{k+j-1}) = (0 \bullet 0, \ldots, \boldsymbol{f_j g_k}, f_{j+1} \bullet 0, \ldots, f_{k+j-1} \bullet 0, 0, \ldots)$
(2) If $k = j$, then $c_{2k} = f_j g_k \neq 0$. That is,
$(0, \ldots, f_j, f_{j+1}, \ldots f_{2k})(0, \ldots, g_k, g_{k+1}, \ldots g_{2k}) = (0 \bullet 0, \ldots, \boldsymbol{f_j g_k}, f_{j+1} \bullet 0, \ldots, f_{2k} \bullet 0, 0, \ldots)$.
In either case, $\exists\ i > 0$ such that $c_i \neq 0$, hence $(c_i) = (f_i)(g_i) = fg \neq 0$.
Thus, by Proposition 3.5 (A nonzero commutative ring R is a domain iff the product of any 2 nonzero elements of R is nonzero.), $R[x]$ is a domain.

Suppose $\deg(g) = s$, $\deg(f) = t$.
So $g_i = 0\ \forall\ i > s$ and $g_s \neq 0$ (i.e. $g = (g_0, g_1, \ldots, g_s, 0, 0, \ldots)$
$f_i = 0\ \forall\ i > t$ and $f_t \neq 0$ (i.e. $f = (f_0, f_1, \ldots, f_t, 0, 0, \ldots)$
Then $(f_i)(g_i) = (c_i)$ where $c_i = \sum_{j+k=i} f_j g_k$. Let's find $c_{s+t} = \sum_{j+k=s+t} f_j g_k$.

Suppose $j > t$, then $f_j = 0$, so $f_j g_j = 0$.
Suppose $j < t$, then $j + k = s + t > s + j$ which implies $k > s$, so $g_k = 0$, and $f_i g_i = 0$.
Suppose $j = t$, then $k = s$, so $f_t g_s \neq 0$ (since $f_t \neq 0$, $g_s \neq 0$ and R is an integral domain).
We can use a similar argument to show $c_r = 0$, $r > s + t$.
We know $c_r = \sum_{j+k=r} f_j g_k$.
Suppose $j > t$, then $f_j = 0$, so $f_j g_j = 0$.
Suppose $j < t$, then $j + k = r > s + t$ which implies $k > s$, so $g_k = 0$, and $f_i g_i = 0$.
Suppose $j = t$, then $j + k = t + k > s + t$ which implies $k > s$, so $f_t g_s \neq 0$ (since $f_t \neq 0$, $g_s \neq 0$ and R is an integral domain).
$\therefore$ $c_{s+t} \neq 0$ and $c_r = 0$ for $r > s + t$.
$\therefore$ $\deg(fg) \leq \deg(g) + \deg(f)$ if $f, g \neq 0$.

3.5 Homomorphisms

Definition Let R, S be rings, then a *ring homomorphism* is a function $\phi: R \to S$ such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ $\forall$ $a, b \in R$.

Note A ring homomorphism is a group homomorphism, so we get all the "group stuff" for free. But be careful to remember this applies on to the addition structure.

Proposition $\phi(a^n) = \phi(a)^n$.

Question Suppose we have $\phi: R \to S$ and R has a unity. What can we say about $\phi(1_R)$? $\phi(1_R)$ is a unity in $\phi(R)$.

Proposition Let $\phi: R \to S$ be an epimorphism (surjective homomorphism)
(1) If R has a unity, then $\phi(1_R)$ is the unity in S.
(2) If $a \in U(R)$, then $\phi(a^{-1}) = \phi(a)^{-1}$.
Proof: Homework exercise.
(1) Since ϕ is surjective, then $\phi(R) = S$.
Let $a \in R$. Then $\phi(a \bullet 1_R) = \phi(a)\phi(1_R)$. So $\phi(1_R)$ is the unity for $\phi(R) = S$.
(2) Let $b \in U(R)$, then $1_S = \phi(1_R) = \phi(b \bullet b^{-1}) = \phi(b)\phi(b^{-1})$. $\therefore$ $\phi(b^{-1}) = \phi(b)^{-1}$.

Example $\phi: \mathbb{Z}_5 \to \mathbb{Z}_{10}$. This is a group homomorphism but not ring homomorphism because $\phi(1)\phi(1) = 2 \bullet 2 = 4$, but $\phi(1) = 2$.

Example $\phi: \mathbb{Z} \to \mathbb{Z}_n$. where $a \mapsto [a] = a[1]$. $\ker \phi = \{a \in \mathbb{Z}: \phi(a) = [0]\} = n\mathbb{Z}$.

Example $\phi: \mathbb{Z} \to R$ where R has a unity and $a \mapsto a \bullet 1_R$. This mean multiples of 1_R. If $\circ(1_R) = n$, then $\ker \phi = n\mathbb{Z}$.

Example $1_R \in R$, $\circ(1_R) = 17$. Let $b \in R$. $17b = ?$

$17b = 17(1_R \bullet b) = \underbrace{1_R \bullet b + \cdots + 1_R \bullet b}_{\text{17 of them}} = (1_R + \cdots + 1_R) \bullet b = (17 \bullet 1_R) \bullet b.$

(The first "17(" is marked *not* mult. in ring; the "•" in $1_R \bullet b$ is marked mult. in ring.)

Definition Let R be a ring, then R has *characteristic* n, if $na = 0\ \forall\ a \in R$ and n is the smallest such positive integer. If no such positive integer exists, then we say char $(R) = 0$.

Note char (R) = $\circ(1_R)$ is R has unity.

Example $\phi: R \to S$. $\phi^*: R[x] \to S[x]$. Then $\phi^*\left(\sum_{i=0}^{\infty} r_i x^i\right) = \sum_{i=0}^{\infty} \Phi(r_i) x^i$.

We can see polynomials in two ways:
$f(x) = x^2 + 5$ can be seen as a polynomial, and
$f(x) = x^2 + 5$ can be seen as a function.

Example Let $f(x) = x^3$ and $g(x) = x$ both in $\mathbb{Z}_3$.
In $\mathbb{Z}_3[x]$, $f(x) \neq g(x)$. (i.e. $(0, 1, 0, \ldots) \neq (0, 0, 0, 1, 0, \ldots)$)
But as functions,
$f(0) = 0 \quad = g(0)$
$f(1) = 1 \quad = g(1)$
$f(2) = 2^3 = 2 \quad = g(2)$
So, as functions, they are equal.

Example Let $a \in R$. $\phi_a : R[x] \to R$ where $\phi_a(f(x)) = f(a)$. This is called the evaluation function.

Example $\phi_3((x + 1)(x - 1)) = \phi_3(x^2 - x - 2) = 3^2 - 3 - 2 = 4$.
$\phi_3(x + 1)\phi_3(x - 2) = (3 + 1)(3 - 2) = 4(1) = 4$.

$\phi_a : R[x] \to R$. $\phi_a((c_1x)(c_2x)) = \phi(c_1c_2x^2) = c_1c_2a^2$. (since R is commutative)
Prove this is a homomorphism as homework.
Proof:

Let $f, g \in R[x]$. Then $\phi_a(f + g) = \Sigma_{i=0}^{\infty}(f_i + g_i)x^i = \Sigma_{i=0}^{\infty} f_i x^i + \Sigma_{i=0}^{\infty} g_i x^i = \phi_a(f) + \phi_a(g)$.
And $\phi_a(fg) = \Sigma_{i=0}^{\infty}(\Sigma_{j+k=i} f_j g_k)x^i = \Sigma_{i=0}^{\infty} f_i x^i \bullet \Sigma_{i=0}^{\infty} g_i x^i = \phi_a(f)\phi_a(g)$.
$\therefore$ ϕ is a ring homomorphism.