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Warm-up How many elements of order 2 are in D_{20}? 11

We know $D_{20} = \langle a,b \mid a^{10} = 1 = b^2; bab^{-1} = a^{-1}\rangle$.
Since $bab = bab^{-1} = a^{-1}$, then $(ab)^2 = (ab)(ab) = (ba^{-1})(ab) = b^2 = 1$.
So we have 10 of $(a^kb)^2$ (since a^k are distinct for $k \in \{1, 2, \ldots, 9\}$) and a^5 has order 2.
Hence we have 11 elements of order 2.

Counterexample For Finite Abelian Groups handout, Exercise #3.

$H_1 = \{(123), (132), (1)\}$
$H_2 = \{(12345), (13524), (14253), (15432), (1)\}$
$H_3 = \{(12)(34), (13)(24), (14)(23), (1)\}$
$\lvert H_1H_2H_3\rvert \neq \lvert H_1\rvert \bullet \lvert H_2\rvert \bullet \lvert H_3\rvert$.
So, the exercise needs to have the added condition that the group, G, is Abelian or that the groups, $H_1, H_2, \ldots, H_k$ be normal to G. Possibly a lesser condition could give the result, but it is not known what that might be.

Example Show no group of order 30 is simple.

Proof:
Let G be a group such that $\lvert G\rvert = 30 = 2 \bullet 3 \bullet 5$.

$n_2 = 1, 3, 5, 15$
$n_3 = 1, 10$
$n_5 = 1, 6$

If any of these is 1, then we're done.

So, suppose $n_2 \neq 1$, $n_3 \neq 1$, and $n_5 \neq 1$.
All Sylow 5-subgroups intersect trivially because $\lvert P_5\rvert = 5$.
If $n_5 = 6$, we have $6 \bullet 4 = 24$ elements of order 5.
If $n_3 = 10$, we have $10 \bullet 2 = 20$ element of order 3.
This gives us more than 30 elements, a contradiction to our assumption that $\lvert G\rvert = 30$.
Thus, it cannot be the case that none of the n_p's are 1.

Example If G is a group such that $\lvert G\rvert = 4125 = 3 \bullet 5^3 \bullet 11$, G is not simple.

Proof:
$n_5 = 1, 11$.
If $n_5 = 1$, then we're done. Assume $n_5 = 11$.
So $11 = [G{:}N_G(P_5)]$ and $\lvert G\rvert \nmid 11!$.
So, by the Index Factorial theorem, G is not simple.

Example If G is a group such that $\lvert G\rvert = 380 = 2^2 \bullet 5 \bullet 19$, G is not simple.

Proof:
$n_2 = 1, 5, 19, 95$ We can throw out 5 by the Index
$n_5 = 1, 76$ Factorial theorem. But that won't
$n_7 = 1, 20$ help us much in this case.

Assume G is simple, then $n_2 \neq 1$, $n_5 \neq 1$, and $n_7 \neq 1$.
So, assume $n_5 = 76$ and $n_7 = 20$. Then G has 76 • 4 = 304 elements of order 5 and 20 • 18 = 360 elements of order 19. Too many elements!
A contradiction.
Thus $n_5 \neq 76$ or $n_7 \neq 20$. $\therefore$ $n_5 = 1$ or $n_7 = 1$. And we're done.

Example What can we say about a group of order 70?
We can say there are exactly 4 non-isomorphic groups of order 70.
Proof:
Let G be a group such that $|G| = 70 = 2 \cdot 5 \cdot 7$.
$n_2 = 1, 5, 7, 35$
$n_5 = 1$
$n_7 = 1$
So, $\exists$ P_5, P_7 such that $P_5 \triangleleft G$ and $P_7 \triangleleft G$.
Since $|P_5| = 5$, and $|P_7| = 7$, then $P_5 \cap P_7 = \{e\}$.
So, by Direct Products Handout Exercise 2 (If H $\triangleleft$ G or K $\triangleleft$ G, HK $\leq$ G.), we have $P_5P_7 \leq G$. And since $P_5 \cong \mathbb{Z}_5$ and $P_7 \cong \mathbb{Z}_7$, then by Finite Abelian Groups Handout, Exercise 1($\mathbb{Z}_m \times \mathbb{Z}_n) \cong \mathbb{Z}_{mn}$ iff $(m, n) = 1$),
then $P_5P_7 \cong P_5 \times P_7$.
So $\exists$ $H \leq G$ such that $H = \langle a \rangle$ where $\circ(a) = 35$.
And by Cauchy's theorem, $\exists$ $b \in G$ such that $\circ(b) = 2$.
So, we know $G = \{1, a, a^2, \ldots, a^{34}, b, ab, a^2b, \ldots, a^{34}b\}$.
If we know how a and b interect, we understand the group.
So, we know $G = \langle a, b \mid \cdots\cdots\text{something}\cdots \rangle$.
Consider bab^{-1}.
$bab^{-1} \in H$ (since $H \triangleleft G$).
We know $H \triangleleft G$ as $[G:H] = 2$ or because $H = P_5P_7 \cong P_5 \times P_7$, the internal direct product; take your pick.
So for $k \in \{1, 2, \ldots, 34\}$, $bab^{-1} = bab = a^k$. So $a = (bab)^k = (a^k)^k = a^{k^2}$.
So $e = a^{k^2-1}$.
And $35 \mid k^2 - 1$, as $\circ(a) = 35$. So $k = 1, 6, 29$, or 34.
Thus, there are at most 4 groups of order 70.
Let's list what we think they are:
$\mathbb{Z}_{70}$, D_{70}, $D_{10} \times \mathbb{Z}_7$, and $D_{14} \times Z_5$.
We know $\mathbb{Z}_{70}$ is not isomorphic to the others as $\mathbb{Z}_{70}$ is cyclic (or Abelian, take your pick) and the others are not.
We know D_{70} has 35 elements of order 2 but $D_{10} \times \mathbb{Z}_7$ has 5 and

$D_{14} \times Z_5$ has 7 elements of order 2.
Thus the groups are all non-isomorphic.
Note $\circ(x, y) = \text{lcm}(\circ(x), \circ(y))$.

Summary of strategies for proving a group is not simple:

- Work with the n_p's.
- Use the Index Factorial theorem. (Use $p^e \nmid (m-1)!$ as a clue to use IFT.)
- Count elements.

- Use Representative on Cosets theorem

Example Let G be a group such that $|G| = 108 = 2^2 \cdot 3^3$.
Show $\exists\, H \lhd G$ such that $|H| = 3^2$ or 3^3.
Proof:
$n_3 = 1$ or 4. If $n_3 = 1$, then $P_3 \lhd G$ and $|P_3| = 3^3$. And we're done.
If $n_3 = 4$, then $[G:N_G(P_3)] = 4$.
By Representation on Cosets theorem, $\exists\, \phi: G \to S_4$ with $\ker \phi \leq N_G(P_3)$.
If $\ker \phi = \{1\}$, then $|G| \mid 4!$, but $|G|$ does not divide $4!$, so $\ker \phi \neq \{1\}$.
Since $\ker \phi \leq N_G(P_3)$ and $|N_G(P_3)| = 27$ (since $4 = 108/|N_G(P_3)|$), then
$|\ker \phi| = 3, 3^2$, or 3^3. Since $\ker \phi \lhd G$, then if $|\ker \phi| = 3^2$ or 3^3, we're done.
So assume $|\ker \phi| = 3$ and start over.
Consider $G/\ker \phi$. $|G/\ker \phi| = 36 = 2^2 \cdot 3^2$.
$n_3 = 1, 4$. If $n_3 = 1$, then $P_3 \lhd G/\ker \phi$
and I'm not sure where to go from here.