Content:
Example Prove that any group of order 108 has a normal subgroup of order 3^2 or 3^3.

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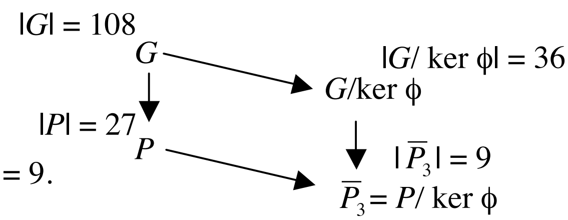
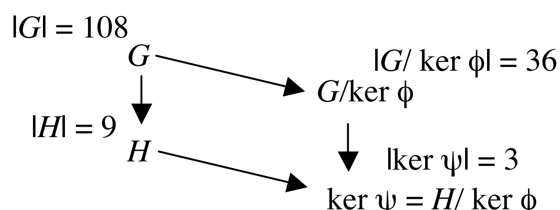
Proof:
Let G be a group such that $|G| = 108$. $108 = 2^2 \bullet 3^3$.
$n_2 = 1, 3, 9, 27$
$n_3 = 1$ or 4.
If $n_3 = 1$, then $P_3 \lhd G$ and $|P_3| = 3^3$. And we're done.
If $n_3 = 4$, then $[G{:}N_G(P_3)] = 4$. This gives us that $|N_G(P_3)| = 27$ (since $4 = 108/|N_G(P_3)|$), hence $N_G(P_3) = P_3$ (since $P_3 \leq N_G(P_3)$ and $|N_G(P_3)| = |P_3|$).
By Representation on Cosets theorem, $\exists\ \phi: G \to S_4$ with $\ker \phi \leq N_G(P_3)$.
Since $|G| \nmid 4!$, then $\ker \phi \neq \{e\}$, by Index Factorial theorem and 1st Isomorphism theorem.
Since $\ker \phi \leq N_G(P_3)$ and $|N_G(P_3)| = 27$ then $|\ker \phi| = 3, 3^2$, or 3^3.
Since $\ker \phi \lhd G$, then if $|\ker \phi| = 3^2$ or 3^3, we're done.
So assume $|\ker \phi| = 3$.
Now consider $G/\ker \phi$.
$|G/\ker \phi| = 36 = 2^2 \bullet 3^2$.
Then $\bar{n}_2 = 1, 3, 9$ and $\bar{n}_3 = 1, 4$.
If $\bar{n}_3 = 1$, then $\exists\ \bar{P}_3 \lhd G/\ker \phi$ and $|\bar{P}_3| = 9$.



Then, by the Correspondence theorem, $\exists\ P \lhd G$ such that $P/\ker \phi = \bar{P}_3$ and $|P| = 27$ (since $|G/\ker \phi|/|\bar{P}_3| = 36/9 = 4 = |G|/|P| = 108/|P|$).
If this is the case, then we're done.
If $\bar{n}_3 = 4$, then $4 = [G/\ker \phi{:}N_{G/\ker\phi}(\bar{P}_3)]$ gives us that $|N_{G/\ker\phi}(\bar{P}_3)| = 9$ ($4 = 36/|N_{G/\ker\phi}(\bar{P}_3)|$).
By Representation on Cosets theorem, $\exists\ \psi: G/\ker \phi \to S_4$ with $\ker \psi \leq N_{G/\ker\phi}(\bar{P}_3)$. Since $|G/\ker \psi| \nmid 4!$, then $\ker \psi \neq \{\ker \phi\}$, by Index Factorial theorem and 1st Isomorphism theorem. Since $\ker \psi \leq N_{G/\ker\phi}(\bar{P}_3)$ and $|N_{G/\ker\phi}(\bar{P}_3)| = 9$ then $|\ker \psi| = 3$ or 3^2.
If $|\ker \psi| = 3$, then by the Correspondence theorem, $\exists\ H \lhd G$ such that $H/\ker \phi = \ker \psi$ and $|H| = 9$ (since $|G/\ker \phi|/|\ker \psi| = 36/3 = 108/|P|$). If $|\ker \psi| = 9$, then by the Correspondence theorem, $\exists\ H' \lhd G$ such that $H'/\ker \phi = \ker \psi$ and $|H'| = 27$ (since $|G/\ker \phi|/|\ker \psi| = 36/9 = 108/|P|$).



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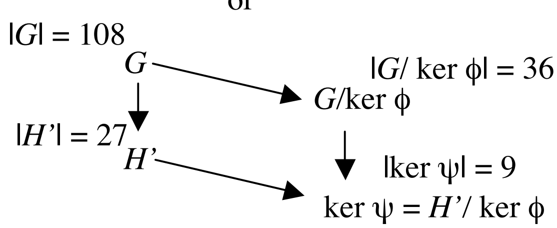

For the test, know

- Correspondence theorem
- Group Action stuff
- Direct Products
- Finite Abelian Groups
- Sylow Theory

Know all your definitions
Lots of examples
Examples to understand the theorems

Discussion regarding homework problem:
5.30 Prove that there is no simple group of order 120.
Proof:
Let G be group such that $|G| = 120 = 2^3 \bullet 3 \bullet 5$.

$n_2 \mid 15 \Rightarrow$ $n_2 = 1, 3, 5, 15$	$n_2 \equiv 1 \pmod 2 \Rightarrow$ $n_2 = 1, 3, 5, 15, \ldots$	$n_2 = 1, 3, 5, 15$
$n_3 \mid 40 \Rightarrow$ $n_3 = 1, 2, 4, 5, 8, 10, 20, 40$	$n_3 \equiv 1 \pmod 3 \Rightarrow$ $n_3 = 1, 4, 7, 10, \ldots, 40$	$n_3 = 1, 4, 7, 10, 40$
$n_5 \mid 24 \Rightarrow$ $n_5 = 1, 2, 3, 6, 12, 24$	$n_5 \equiv 1 \pmod 5 \Rightarrow$ $n_5 = 1, 6, 11, 16, \ldots$	$n_5 = 1, 6$

Assume G is simple. Then $n_2 \neq 1$, $n_3 \neq 1$, and $n_5 \neq 1$, hence $n_5 = 6$.
Then, by Representation on Cosets, $\exists\ \phi: G \to S_6$ where $\ker \phi \leq N_G(P_5)$.
Since G is simple by assumption, then $\ker \phi = \{e\}$. So $G \cong \phi(G) \leq S_6$.
Notice that $\phi(G) \cap A_6 \triangleleft \phi(G)$ (by 2nd Isomorphism theorem).
And by Exam 1, # 7(c), $|\phi(G) \cap A_6| = |\phi(G)| = 120$ or $|\phi(G) \cap A_6| = (1/2)|\phi(G)| = 60$.
If $\sim(\phi(G) \leq A_6)$, then $\phi(G) \cap A_6 \neq \phi(G)$, hence $|\phi(G) \cap A_6| = (1/2)|\phi(G)| = 60$.
But since $\phi(G) \cap A_6 \triangleleft A_6$, and A_6 is simple, and $|\phi(G) \cap A_6| \neq 1$ then we have a contradiction. (But, how do we know $\phi(G) \cap A_6 \triangleleft A_6$?)
If $\phi(G) \leq A_6$, then $\phi(G) \cap A_6 = \phi(G)$, hence $|\phi(G) \cap A_6| = |\phi(G)| = 120$.
So $[A_6 : \phi(G) \cap A_6] = 360/120 = 3$.
Then by the Representation on Cosets theorem,
$\exists\ \psi : A_6 \to S_3$ where $\ker \psi \leq A_6/\phi(G) \cap A_6$. And since A_6 is simple, then $\ker \psi = \{(1)\}$.
Thus, by the 1st Isomorphism theorem, $A_6 \cong \psi(A_6)$.
But $\psi(A_6) \leq S_3$ and $|A_6| = 120 > 6 = |S_3|$.
So $\ker \phi \neq \{e\}$, hence G is not simple.

Discussion on Index Factorial theorem(s)

(1) If $H \leq G$ such that $[G:N] = n$ and $|G| \nmid n!$, then G is not simple.
(2) $p^k \nmid (m-1)!$
(3) $|G| \nmid n_p!$

The 2nd two are just special cases of the first. We can use them on an exam in this case, be would not be able to anywhere outside of class (i.e. the comps).
Good luck on your test everyone!