Content:

Proposition	Frac (R) contains an isomorphic copy of R
Definition	$F(x)$
Proposition	Let k be a field containing an integral domain R. Then there exists an isomorphic copy of Frac (R) in k. (i.e. Frac (R) is the smallest field containing R.)

Recall

R Integral Domain
F = Frac (R)
$[a, b] = 0_F \Leftrightarrow b \neq 0, a = 0$.
$[a, a] = 1_F$
$[a, b]^{-1} = [b, a]$ if $a \neq 0$.
F is a field.

Example $\text{Frac}(\mathbb{Z}) \cong \mathbb{Q}$. $\mathbb{Q}[a, b] \mapsto a/b$

Proposition Frac (R) contains an isomorphic copy of R.

Proof:
Define $\varphi: R \to \text{Frac}(R)$ by $\varphi(r) = [r, 1]$. ← --if we didn't have unity, we would choose $[rb, b]$.
Define $R' = \{[a, 1]: a \in R\}$
It's easy to see $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$.
Check this on your own.
$\varphi(a + b) = [a + b, 1] = [a, 1] + [b, 1] = \varphi(a) + \varphi(b)$ and
$\varphi(ab) = [ab, 1] = [a, 1][b, 1] = \varphi(a)\varphi(b)$.
Let $a \in \ker \varphi$. Then $\varphi(a) = 0_{R'}$. So $[a, 1] = 0_{R'}$. Thus $a = 0$. So φ is 1-1.

In Frac (R) elements look like $[a, b] = a/b = ab^{-1}$ (be careful about ab^{-1})

Example Let F be a field. Then

$$\begin{aligned}\text{Frac}(F[x]) &= \{[f(x), g(x)] \mid f, g \in F[x], g \neq 0\} \\ &= \{f(x)/g(x) \mid f, g \in F[x], g \neq 0\} \\ &= F(x) \\ &= \text{Field of Rational Functions}\end{aligned}$$

In homework exercises will prove $\text{Frac}(R[x]) \cong F(x)$.

Example Suppose k is a field. What is Frac (k)? (Hopefully a field)

$$\begin{aligned}\text{Frac}(k) &= \{[a, b] \mid a, b \in k, b \neq 0\} \\ &= \{[ab^{-1}, 1] \mid a, b \in k, b \neq 0\}\end{aligned}$$

$$[a, b] = [ab^{-1}, 1] \text{ since } a \bullet 1 = b(ab^{-1})$$

Define $\varphi: k \to \text{Frac}(k)$ by $\varphi(a) = [a, 1]$.
From previous proposition, we know φ is an injective homomorphism.
Let $[a, b] \in \text{Frac}(k)$, then $\varphi(ab^{-1}) = [ab^{-1}, 1] = [a, b]$.
So φ is onto and hence an isomorphism.

Note Frac(Frac (R)) $\cong$ Frac (R).

Proposition Let k be a field containing an integral domain R. Then there exists an isomorphic copy of Frac (R) in k. (i.e. Frac (R) is the smallest field containing R.)

Proof:
Define Frac (R) $\to k$ by $\varphi([a, b]) = ab^{-1}$.
$\varphi([a, b] + [c, d]) = \varphi[ad + bc, bd] = (ad + bc)(bd)^{-1} = ad(bd)^{-1} + bc(bd)^{-1} = ab^{-1} + cd^{-1} = \varphi([a, b]) + \varphi([c, d])$. And
$\varphi([a, b][c, d]) = \varphi[ac, bd] = (ac)(bd)^{-1} = ab^{-1} \bullet cd^{-1} = \varphi([a, b])\varphi([c, d])$.
Check well-defined and 1-1:
Suppose $[a, b] = [c, d]$ ($[a, b], [c, d] \in$ Frac (R); $a, b, c, d \in R \subseteq k$).
Then $ad = bc$
And $ab^{-1} = cd^{-1}$ (since $a, b, c, d \in k$).
So $\varphi[a, b] = \varphi([c, d])$.
The reverse of this gives us 1-1.
Or, we could have checked 1-1 by looking at the kernel.
$\therefore$ Frac (R) $\cong \varphi($ Frac (R)) $\subseteq k$.

Note Since the kernel of a field is {0} or all of the field, then:
If checking φ: Frac (R) $\to R$ for 1-1, it's easier to look at the kernel.
If checking φ: $R \to$ Frac (R) for well-defined it's easier to look at the kernel. Or, it might be the other way around.

Example Let $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} = \mathbb{Z}[\sqrt{2}]$.
Frac (R) $\cong$ (?) We guess $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.
Options
(1) Define φ: Frac (R) $\to \mathbb{Q}(\sqrt{2})$ and show it's an isomorphism.
(2) Show $\mathbb{Q}(\sqrt{2})$ is the smallest field containing R.
We'll choose **(2).**
We'll show a) $\mathbb{Q}(\sqrt{2})$ is a field that contains R.
b) If $R \subseteq k$, k is a field, then $\mathbb{Q}(\sqrt{2}) \subseteq k$.
(a) Since $\mathbb{Q}(\sqrt{2}) \subseteq$ R, we need only check closure of +, •, and that we have inverses of nonzero elements.
Closure – you check.
$a + b\sqrt{2} + c + d\sqrt{2} = a + c + (b + d)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$
$(a + b\sqrt{2})(c + d\sqrt{2}) = ac + 2bd + (cb + ad)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$
$(a + b\sqrt{2})^{-1} = ?$ ($a, b \in \mathbb{Q}$, not both 0).
Since $(a + b\sqrt{2})\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\sqrt{2}\right) = 1$, then

$$(a + b\sqrt{2})^{-1} = \left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\sqrt{2}\right).$$

So we have (a)

(b) Let k be a field such that $R \subseteq k$. We'll show $\mathbb{Q}(\sqrt{2}) \subseteq k$.

Let $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$.

Then $a = p/q$ and $b = m/n$ for some $p, q, m, n \in \mathbb{Z}$, $q \neq 0$ and $n \neq 0$.

We know $\mathbb{Z} \subseteq R \subseteq k$, and $\mathbb{Q} \cong \text{Frac}(\mathbb{Z}) \subseteq k$ (since Frac ($\mathbb{Z}$) is the smallest field containing $\mathbb{Z}$.) Since $p/q, m/n \in \text{Frac}(\mathbb{Z})$, and $\sqrt{2} \in k$. then

$a + b\sqrt{2} = p/q + (m/n)\sqrt{2} \in k$.

So $\mathbb{Q}(\sqrt{2}) \subseteq k$. Hence $\mathbb{Q}(\sqrt{2})$ is the smallest field containing R.
