Debra Griffin
Section 3.7
Worksheet #1 - 13

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1. Lemma 7. *Let V be a vector space over F. Let $U \subseteq V$. Then U is a subspace iff (1) $U \neq \emptyset$; (2) $u + v \in U$ $\forall u, v \in U$; (3) $au \in U$ $\forall a \in F, u \in U$.*

Proof:
To prove the forward implication assume U is a subspace.
Then we have that U is an Abelian group, hence $0 \in U$, and so $U \neq \emptyset$.
If $u, v \in U$, then by additive closure of the Abelian group U, $u + v \in U$.
If $a \in R$ and $u \in U$, then as U is a vector space over F, we have $au \in U$.

To prove the converse, assume $U \subseteq V$ and the three properties hold.
Let $u \in U$. We know u exists by (1) $U \neq \emptyset$.
Since $0_F \in F$, then $0_F \bullet u \in U$ by (3).
And since $u \in V$ and V is a vector space over F, then $0_F \bullet u = 0_V$. Thus $0_V \in U$.
Similarly $-1_F \bullet u \in -u \in U$.
So then if $u, v \in U$, we have $-v \in U$ and (2) gives us that $u - v \in U$.
Thus $U \leq V$. Since $U \subseteq V$, then U inherits its operations from V.
$\therefore$ U is a subspace of V.

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2. *In $\mathbb{R}[x]$ consider the set $V = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{R}\}$.*
(a) *Prove that V is a subspace of $\mathbb{R}[x]$.*

Proof:
We have $0 = 0x^2 + 0x + 0 \in V$, thus $V \neq \emptyset$.
If $u, v \in V$, then for some $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$,
$u + v = a_2x^2 + a_1x + a_0 + b_2x^2 + b_1x + b_0 = (a_2 + b_2)x^2 + (a_1 + b_1)x + a_0 + b_0 \in V$
as $a_2 + b_2, a_1 + b_1, a_0 + b_0 \in \mathbb{R}$.
And if $r \in \mathbb{R}$ and $u = a_2x^2 + a_1x + a_0 \in V$, then $ru = ra_2x^2 + ra_1x + ra_0 \in V$
as $ra_2, ra_1, ra_0 \in \mathbb{R}$. $\therefore$ V is a subspace of $\mathbb{R}[x]$.

(b) *Find a basis for V.*
A basis for V is $\{1, x, x^2\}$.
Proof:
Let $u \in V$, then $u = a_2 \bullet x^2 + a_1 \bullet x + a_0 \bullet 1$ for some $a_0, a_1, a_2 \in \mathbb{R}$.
$\therefore$ u is a linear combination of $1, x, x^2$.
If $0 = a_2 \bullet x^2 + a_1 \bullet x + a_0 \bullet 1$, then $a_2 = a_1 = a_0$ by definition of polynomial,
hence $\{1, x, x^2\}$ is linearly independent.
$\therefore$ $\{1, x, x^2\}$ is a basis for V.

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2. (c) *Is $\{x^2 + x + 1, x + 5, 3\}$ a basis?*

Yes.

Proof:

Let $u = ax^2 + bx + c \in$ span V.

Then $u = a(x^2 + x + 1) + (b - a)(x + 5) + \frac{1}{3}(c - 5b + 5a) \bullet 3$.

$\therefore$ $\{x^2 + x + 1, x + 5, 3\}$ spans V.

And if $0 = a(x^2 + x + 1) + (b - a)(x + 5) + \frac{1}{3}(c - 5b + 5a) \bullet 3$, then

$a = 0$, hence $b = 0$, hence $c = 0$.

$\therefore$ $\{x^2 + x + 1, x + 5, 3\}$ is linearly independent over V.

Thus $\{x^2 + x + 1, x + 5, 3\}$ is a basis for V.

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3. *Let $V = \mathbb{R}^3$ and let $W = \{(a, b, c) \in \mathbb{R}^3: a^2 + b^2 = c^2\}$. Is W a subspace of V? If so, what is its dimension?*

No, W is not a subspace of V.

Proof:

If (a, b, c) and $(a', b', c') \in W$, then $a^2 + b^2 = c^2$ and $a'^2 + b'^2 = c'^2$.

However, $(a, b, c) + (a', b', c') = (a + a', b + b', c + c')$ but

$(a + a')^2 + (b + b')^2 = a^2 + 2aa' + a'^2 + b^2 + 2bb' + b'^2 = c^2 + c'^2 + 2aa' + 2bb'$

$\neq c^2 + 2cc' + c'^2 = (c + c')^2$. Thus W lacks closure under addition.

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4. *Let $V = \mathbb{R}^3$ and let $W = \{(a, b, c) \in \mathbb{R}^3: a + b = c\}$. Is W a subspace of V? If so, what is its dimension?*

Yes, W is a subspace of V and dim $W = 2$.

Proof:

$0 + 0 = 0$, so $(0, 0, 0) \in W$, hence $W \neq \emptyset$.

If (a, b, c) and $(a', b', c') \in W$, then $a + b = c$ and $a' + b' = c'$. So $a + a' + b + b' = c + c'$. Thus $(a, b, c) + (a', b', c') = (a + a', b + b', c + c') \in W$.

If $r \in \mathbb{R}$, and $(a, b, c) \in W$, then $r(a, b, c) = (ra, rb, rc) \in W$ as $ra + rb = r(a + b) = rc$.

Thus, W is a subspace of V.

Claim: $\{(1, 0, 1), (0, 1, 1)\}$ is a basis for W.

Let $w \in W$, then $w = (a, b, c)$ for some $a, b, c \in$ R such that $a + b = c$.

Since $(a, b, c) = (a, b, a + b) = (a, 0, a) + (0, b, b) = a(1, 0, 1) + b(0, 1, 1)$, then

$w \in$ span $\{(1, 0, 1), (0, 1, 1)\}$. Thus, $\{(1, 0, 1), (0, 1, 1)\}$ is a basis for W.

$\therefore$ dim $W = 2$.

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5. *Prove that every spanning set contains a basis: If $\{v_1, v_2, \ldots, v_n\}$ is a spanning set for a vector space V, then some subset of V is a basis for V.*

Proof:

If $v_1, \ldots, v_n$ are linearly independent, then $\{v_1, \ldots, v_n\}$ is a basis for V, and we're done.

If not, then there are $a_i \in F$, not all 0, such that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$.

Without loss of generality assume $a_1 \neq 0$. Then $v_1 = a_1^{-1}(-a_2v_2 - \cdots - a_nv_n)$.

Thus, span$\{v_1, \ldots, v_n\}$ = span$\{v_2, \ldots, v_n\}$.

Since a single nonzero vector is linearly independent, then we can continue in this manner deleting v_i's until the remaining elements are linearly independent.

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6. *Prove that every linearly independent set can be extended to a basis:*
If $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set in a vector space V, then there exist vectors, $w_1, w_2, \ldots, w_m$ in V such that $\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m\}$ is a basis for V.

Proof:
If span $\{v_1, \ldots, v_n\} = V$, then $B = \{v_1, \ldots, v_n\}$ and we're done.
If not, $\exists\, w_1 \in V$ such that $w_1 \notin$ span $\{v_1, \ldots, v_n\}$.
Claim: $\{v_1, \ldots, v_n, w_1\}$ is a linearly independent set.
Suppose $a_1v_1 + a_2v_2 + \cdots + a_nv_n + b_1w_1 = 0$.
If $b_1 \neq 0$, then $w_1 = b_1^{-1}(-a_1v_1 - a_2v_2 - \cdots - a_nv_n)$.
But this contradicts that $w_1 \notin$ span $\{v_1, \ldots, v_n\}$ by assumption.
So $b_1 = 0$. And by linear independence of $v_1, \ldots, v_n$, all $a_i = 0$, hence $\{v_1, \ldots, v_n, w_1\}$ is a linearly independent set.
If span $\{v_1, \ldots, v_n, w_1\} = V$, then $B = \{v_1, \ldots, v_n, w_1\}$ are we're done.
Since V spans V, then we can repeat the procedure until span $\{v_1, \ldots, w_1, w_2, \ldots\} = V$ or $B = V$.

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7. If V is a vector space over F of dimension 5 and U and W are subspaces of V of dimension 3, prove that $U \cap W \neq \{0\}$.

Proof:
Let span $\{u_1, u_2, u_3\} = U$, and span $\{w_1, w_2, w_3\} = W$.
Then by exercise #6 above, dim $V = 5$ implies $\{u_1, u_2, u_3, w_1, w_2, w_3\}$ is a linearly dependent (otherwise there is a basis B for V that contains $\{u_1, u_2, u_3, w_1, w_2, w_3\}$, which would force dim $V \geq 6$).
So, one of the vectors of $\{u_1, u_2, u_3, w_1, w_2, w_3\}$ is a linear combination of the others. Without loss of generality, assume it's u_1.
Then $u_1 = a_2u_2 + a_3u_3 + b_1w_1 + b_2w_2 + b_3w_3$. So $u_1 - a_2u_2 - a_3u_3 = b_1w_1 + b_2w_2 + b_3w_3$.
We know $0 \neq u_1 - a_2u_2 - a_3u_3$ as then $u_1 = a_2u_2 + a_3u_3$, which contradicts linear independence of $\{u_1, u_2, u_3\}$.
And since $u_1 - a_2u_2 - a_3u_3 \in U$ and $b_1w_1 + b_2w_2 + b_3w_3 \in W$, then we have a nonzero element in the intersection of U and W.
$\therefore U \cap W \neq \{0\}$.

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8. *Let $\{v_1, v_2, \ldots, v_n\}$ be a finite set of vectors in a vector space V over F. Prove that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V if and only if every member of V can be written uniquely as a linear combination of the vectors in $\{v_1, v_2, \ldots, v_n\}$.*

Proof:
Assume $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, and suppose for some $u \in V$ that
$u = a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n$. Then
$0 = b_1v_1 + b_2v_2 + \cdots + b_nv_n - (a_1v_1 + a_2v_2 + \cdots + a_nv_n) = (b_1 - a_1)v_1 + (b_2 - a_2)v_2 + \cdots + (b_n - a_n)v_n$. Since $\{v_1, v_2, \ldots, v_n\}$ is a basis, then each $b_i - a_i = 0$, hence $b_i = a_i$ for each i.
Consequently, u can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$.

Conversely, assume every member of V can be written uniquely as a linear combination of the vectors in $\{v_1, v_2, \ldots, v_n\}$. Then $\{v_1, v_2, \ldots, v_n\}$ spans V and we only need to show $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. Suppose it is not.
Then for some i, v_i is a linear combination of the other vectors in $\{v_1, v_2, \ldots, v_n\}$.
Without loss of generality, assume it is v_1. Then $v_1 = a_2v_2 + \cdots + a_nv_n$ for some $a_i \in F$.
Note that $0 \notin \{v_1, v_2, \ldots, v_n\}$ as $a \bullet 0 = b \bullet 0$ $\forall$ $a, b \in F$ such that $a \neq b$, a contradiction to unique representation of elements of V, so $v_1 \neq 0$.
And so $0 \neq v_1 = v_1 + 0v_2 + 0v_3 + \cdots + 0v_n = 0v_1 + a_2v_2 + \cdots + a_nv_n$ are two distinct linear combinations of $v_1, v_2, \ldots, v_n$, a contradiction to our assumption.
$\therefore$ $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, hence $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

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9. *Let V be a vector space over $\mathbb{Z}_p$ with dim V = n. How many elements are in V? (Hint: Use #8).*

Proof:
Let p be a prime number and $\{v_1, v_2, \ldots, v_n\}$ be a basis for V over $\mathbb{Z}_p$. Then every element of V can be written as a linear combination of $v_1, v_2, \ldots, v_n$.
Note that $[a]_p = [b]_p$ iff $a \equiv b \pmod{p}$. And so if b is an integer, there are exactly p distinct elements $[a]_p \in \mathbb{Z}_p$ such that $[a]_p \neq [b]_p$ since p is prime.
And since each element of V can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$, then we have exactly p possible coefficients for each vector, hence p^n possible linear combinations of $v_1, v_2, \ldots, v_n$. Thus, there are p^n elements in V.

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10. *Let V be a finite dimensional vector space over F. Let U be a subspace of V.*
(a) *Prove dim U ≤ dim V.*

Proof:
Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Let $\{w_1, w_2, \ldots, w_m\}$ be a basis for U.
Suppose $m > n$, then by exercise #6 above, linear independence of $\{w_1, w_2, \ldots, w_m\}$ and $U \subseteq V$, give us that there is a basis B of V, such that $\{w_1, w_2, \ldots, w_m\} \subseteq B$. So B has more than n vectors. But by Theorem 16 (If V is a vector space over F, and $X = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then every basis for V has n vectors.) we have that any basis of V has n vectors, a contradiction.
So $m \le n$, hence dim $U \le$ dim V, as desired.

(b) *Prove that dim U < dim V, if U ≠ V.*

Proof:
Assume $U \neq V$. If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V and $\{w_1, w_2, \ldots, w_m\}$ is a basis for U, then by part (a) we have that $m \le n$.
Since $U \neq V$ and $U \subset V$, then $\exists\, v \in V$ such that $v \notin U$.
Thus $v \neq a_1w_1 + a_2w_2 + \cdots + a_mw_m$ for any $a_i \in F$ (otherwise $v \in U$).
Consequently $\{w_1, w_2, \ldots, w_m\}$ does not span V.
Then by exercise #6 above and linear independence of $\{w_1, w_2, \ldots, w_m\}$, there is a basis for V that contains $\{w_1, w_2, \ldots, w_m\}$ which would force dim $V > m$, a contradiction to the result of part (a).
$\therefore$ $m < n$, hence dim $U <$ dim V.

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11. *Prove Theorem 15. Let $\{v_1, v_2, v_3, \ldots\}$ be an infinite basis for a vector space V. Then any basis for V will be infinite.*

Proof:
Suppose $\{w_1, \ldots, w_n\}$ is a basis for V.
Then $w_1 = a_{1_1} v_1 + a_{2_1} v_2 + \cdots + a_{m_1} v_{m_1}$
where m_1 is the largest index of the a_{i_1}'s such that $a_{i_1} \neq 0$.
And $w_2 = a_{1_2} v_1 + a_{2_2} v_2 + \cdots + a_{m_2} v_{m_2}$ where
where m_2 is the largest index of the a_{i_2}'s such that $a_{i_2} \neq 0$.
Let m_r be the largest index of the a_{i_r}'s such that $a_{i_r} \neq 0$ for at least one of the linear combinations for $w_1, \ldots, w_n$.
Let $t > m_r$. Then since $\{w_1, \ldots, w_n\}$ is a basis for V,
v_t is a linear combination of $w_1, \ldots, w_n$, hence is a linear combination of $v_1, v_2, v_3, \ldots v_{m_r}$.
Thus $v_t = a_1v_1 + \cdots + a_{m_r} v_{m_r}$ for some $a_i \in F$.
But then $0 = -1_F v_t + a_1v_1 + \cdots + a_{m_r} v_{m_r}$ and not all a_i's are 0, namely -1_F,
a contradiction to linear independence of $\{v_1, v_2, v_3, \ldots\}$.
Thus any basis for V must be infinite.

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12. *Let T be a linear transformation of V onto W (where V and W are both vector spaces over a field F). If $\{v_1, v_2, \ldots, v_n\}$ spans V, show that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans W.*

Proof:
Let V and W be vector spaces over a field F.
Let $T: V \to W$ be a linear transformation of V onto W.
Let $\{v_1, v_2, \ldots, v_n\}$ span V.
Let $w \in W$. Since T is mapped onto W, then $\exists\, v \in V$ such that $T(v) = w$.
Then $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ for some $a_i \in F$.
Note that $a_1v_1, a_2v_2, \ldots, a_nv_n \in V$ as each is the linear combination
$0v_1 + 0v_2 + \cdots 0v_{i-1} + a_iv_i + 0v_{i+1} + \cdots + 0v_n$ for some i.
And so $w = T(v) = T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = T(a_1v_1) + T(a_2v_2) + \cdots + T(a_nv_n) = a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n)$. Since W is a vector space over F and $a_1, a_2, \ldots, a_n \in F$, then w is a linear combination of $T(v_1), T(v_2), \ldots, T(v_n)$.
Thus $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans W.

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13. *Let V be a vector space over a field F with dim V = n.*
Prove that V is isomorphic to F^n.
Proof:
Let $\{v_1, \ldots, v_n\}$ be a basis for V. Let $u \in V$, then $u = a_1v_1 + \cdots + a_nv_n$ for some $a_i \in F$.
Define $T: V \to F^n$ by $T(u) = (a_1, a_2, \ldots, a_n)$.
We will show T is a vector space isomorphism.

To show $T(u + v) = T(u) + T(v)$, let
$u = a_1v_1 + a_2v_2 \cdots + a_nv_n$ and $v = b_1v_1 + b_2v_2 \cdots + b_nv_n$ for some $a_i, b_i \in F$.
Then $T(u + v) = T((a_1 + b_1)v_1 + (a_2 + b_2)v_2 \cdots + (a_n + b_n)v_n)$
$= ((a_1 + b_1), (a_2 + b_2)\ldots,(a_n + b_n)) = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = T(u) + T(v)$.

To show $T(au) = aT(u)$ for any $a \in F$ and for all $u \in V$, let $a \in F$, and define u as above. Then $T(au) = T(aa_1v_1 + \cdots + aa_nv_n) = (aa_1, aa_2, \ldots, aa_n)$
$= a(a_1, a_2, \ldots, a_n) = aT(u)$.

To show T is well-defined, assume $T(u) \neq T(v)$ for some $u, v \in V$.
Then $(a_1, a_2, \ldots, a_n) \neq (b_1, b_2, \ldots, b_n)$. So $\exists\, i$ such that $a_i \neq b_i$.
And since each element of V is uniquely written as a linear combination of $\{v_1, \ldots, v_n\}$, then $a_1v_1 + a_2v_2 \cdots + a_nv_n \neq b_1v_1 + b_2v_2 \cdots + b_nv_n$. $\therefore$ T is well-defined.

To show T is surjective, let $(a_1, a_2, \ldots, a_n) \in F^n$, then since $a_1, a_2, \ldots, a_n \in F$, we have $T(a_1v_1 + \cdots + a_nv_n) = (a_1, a_2, \ldots, a_n)$ where $a_1v_1 + \cdots + a_nv_n \in V$. $\therefore$ T is surjective.

And lastly, to show T is injective, assume $T(u) = T(v)$ for some $u, v \in V$.
Then $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$. So for each i we have $a_i = b_i$.
Thus, $u = a_1v_1 + a_2v_2 \cdots + a_nv_n = b_1v_1 + b_2v_2 \cdots + b_nv_n = v$. $\therefore$ T is injective.

$\therefore$ T is a vector space isomorphism.
