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Exercise 8(b/c) Determine the lattice structure for the subgroups of $\mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q})$ and the corresponding subfields of $\mathbb{Q}(\sqrt[4]{2}, i)$.

Exercise 8 Let $E = \mathbb{Q}(\sqrt[4]{2}, i)$.

Consider the following automorphisms on E.

		a_1	a_2	a_3	a_4	
$r^4 = s^2$	Id.	$\sqrt[4]{2}$	$-\sqrt[4]{2}$	$\sqrt[4]{2}\,i$	$-\sqrt[4]{2}\,i$	(1)
r^2	$a_1 \mapsto a_2$ $i \mapsto i$	$-\sqrt[4]{2}$	$\sqrt[4]{2}$	$-\sqrt[4]{2}\,i$	$\sqrt[4]{2}\,i$	(12)(34)
r	$a_1 \mapsto a_3$ $i \mapsto i$	$\sqrt[4]{2}\,i$	$-\sqrt[4]{2}\,i$	$-\sqrt[4]{2}$	$\sqrt[4]{2}$	(1324)
r^{-1}	$a_1 \mapsto a_4$ $i \mapsto i$	$-\sqrt[4]{2}\,i$	$\sqrt[4]{2}\,i$	$\sqrt[4]{2}$	$-\sqrt[4]{2}$	(1423)
s	$a_1 \mapsto a_1$ $i \mapsto -i$	$\sqrt[4]{2}$	$-\sqrt[4]{2}$	$-\sqrt[4]{2}\,i$	$\sqrt[4]{2}\,i$	(34)
$r^2 s$	$a_1 \mapsto a_2$ $i \mapsto -i$	$-\sqrt[4]{2}$	$\sqrt[4]{2}$	$\sqrt[4]{2}\,i$	$-\sqrt[4]{2}\,i$	(12)
rs	$a_1 \mapsto a_3$ $i \mapsto -i$	$\sqrt[4]{2}\,i$	$-\sqrt[4]{2}\,i$	$\sqrt[4]{2}$	$-\sqrt[4]{2}$	(13)(24)
$r^3 s$	$a_1 \mapsto a_4$ $i \mapsto -i$	$-\sqrt[4]{2}\,i$	$\sqrt[4]{2}\,i$	$-\sqrt[4]{2}$	$\sqrt[4]{2}$	(14)(23)

Since s does not fix i, then $\mathbb{Q}(i)$ is the fixed field for $\langle r\rangle$.

$\mathbb{Q}(\sqrt[4]{2})$ is the fixed field for $\langle s\rangle$ as $[D_8:\langle s\rangle] = [\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}] = 4$.

See homework for complete list of dimensions and degrees.

To find the fixed field for rs, let $\alpha \in E^{rs}$. Then

$\alpha = c_0 + c_1\sqrt[4]{2} + c_2\sqrt[4]{2}^{\,2} + c_3\sqrt[4]{2}^{\,3} + c_4 i\sqrt[4]{2} + c_5\sqrt[4]{2} + c_6 i\sqrt[4]{2}^{\,2} + c_7 i\sqrt[4]{2}^{\,3}$ for some $c_i \in \mathbb{Q}$.

Then $\alpha = r(s(\alpha))$

$= c_0 + c_1\sqrt[4]{2}\,i + c_2(\sqrt[4]{2}\,i)^2 + c_3(\sqrt[4]{2}\,i)^3 + c_4(-i) + c_5(-i)\sqrt[4]{2}\,i + c_6(-i)(\sqrt[4]{2}\,i)^2 + c_7(-i)(\sqrt[4]{2}\,i)^3$

$= c_0 + c_1\sqrt[4]{2}\,i - c_2(\sqrt[4]{2})^2 - c_3(\sqrt[4]{2})^3 i - c_4 i + c_5\sqrt[4]{2} + c_6 i(\sqrt[4]{2})^2 - c_7(\sqrt[4]{2})^3$.

Thus, $c_2 = -c_2, \Rightarrow c_2 = 0$, $c_4 = -c_4 \Rightarrow c_4 = 0$, $c_5 = c_5$, $c_3 = -c_7$.

So we can let $c_1 = c_5 = 1$ and all the rest 0. Thus, $\alpha = \sqrt[4]{2} + i\sqrt[4]{2}$.

Then rs fixes $\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2})$.

For a quick guess at the last field, try the conjugate of $\sqrt[4]{2} + i\sqrt[4]{2}$.

Then we have the following lattice structures: