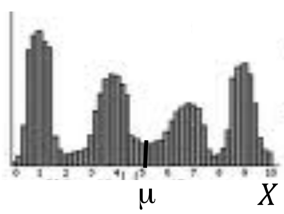
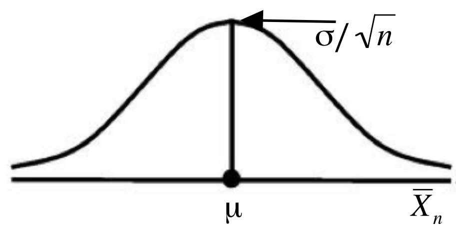
Recall:
1. Given iid's X_i (i = 1, 2, ..., n) with $M_{X_i}(t) = M(t)$, then for $Y = X_1 + \cdots + X_n$, $M_Y(t) = [M(t)]^n$.

2. Sampling distribution of estimates:
$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n}$ is used to estimate μ = $E(X)$, and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is used to estimate σ^2 = Var(X).
$E(\overline{X}_n) = \mu$, $E(S_n^2) = \sigma^2$.
Var($\overline{X}_n$) = σ^2/n.
Var(S_n^2) = (?). It depends on the distribution of X, "difficult".
Even though we know E($\overline{X}_n$) and Var($\overline{X}_n$), we usually don't know the exact distribution of $\overline{X}_n$.





Good news: When n is large, the distribution of $\overline{X}_n$ is approximately normal (central limit theorem). That is, $\overline{X}_n \sim N(\mu, \sigma^2/n)$ approximately.

Theoretical results:

mgf of $\overline{X}_n$: $M_{\overline{X}_n}(t) = E(e^{t\overline{X}_n}) = E(e^{\frac{t}{n}Y}) = M_Y(t/n) = (M(t/n))^n$. "Good result."

Examples:

1. For $X_i \sim N(\mu, \sigma^2)$ (i = 1, 2, ..., n), $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Rightarrow M_{\overline{X}_n}(t) = M_Y(t/n) = (M(t/n))^n =$
$\left(e^{\mu\frac{t}{n} + \frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2}\right)^n = e^{\mu t + \frac{1}{2}\frac{\sigma^2}{n}t^2}$ = mgf of $N(\mu, \sigma^2/n)$. **Good for exam.**

2. $X_i \sim \Gamma(\lambda, \alpha)$. Special case: Let $T \sim$ Exp(λ). $M_T(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \lambda\int_0^\infty e^{-(\lambda - t)x}dx = \frac{\lambda}{\lambda - t} = \frac{1}{1 - \frac{t}{\lambda}}$ for $t < \lambda$. When $\mathbb{Z}^+$, if $X \sim \Gamma(\lambda, \alpha)$ distribution, then X is considered to be a sum of α RV's, $T_1, T_2, \ldots, T_n$ such that $T_i \sim$ Exp(λ) (i = 1, 2, ..., n) and $\{T_1, T_2, \ldots, T_n\}$ are iid's.
Can do the same thing with chi² with n degrees of freedom, independent normal distribution. $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$ when $t < \lambda$.

Averages of X_i's:

In general ($\alpha > 0$) $X \sim \Gamma(\lambda, \alpha)$ implies $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$ for $t < \alpha\lambda$.

Use the pdf of $\Gamma(\lambda, \alpha)$ to verify this result. Use geometric distribution.

Now consider an iid sequence of n RV's $X_1, X_2, \ldots, X_n$ with $\Gamma(\lambda, \alpha)$ distribution. Find the mgf of $\overline{X}_n$ and show it is also a gamma distribution (find the parameters). "Simple exercise."

Markov's (Chebyshev's) Inequality

Let X be a random variable with mean $\mu = E(X)$ and $g(x)$ a non-negative function. Then for $r > 0$, $P(g(X) \geq r) \leq E(g(X))/r$ (assuming $E(X)$ exists).

Proof:

$0 \leq I(g(X) \geq r) \bullet r \leq g(X)$ with probability 1.

Take expected value of both sides.

$r \bullet E(E(g(X) \geq r)) \leq E(g(X))$

$P(g(X) \geq r)$

For the final:

Know the properties of normal distribution and chi^2 distribution.

Final will have a lot of computation of joint pdf, transformation.

Material will cover from take-home to present.

A sequence of random variables $X_1, X_2, \ldots$, converges to X in probability if $\forall\ \varepsilon > 0$, $\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$ or $\lim_{n \to \infty} P(|X_n - X| < \varepsilon) = 1$.

Weak Law of Large Numbers

$\overline{X}_n \to \mu$ in probability where $\mu = E(X)$.