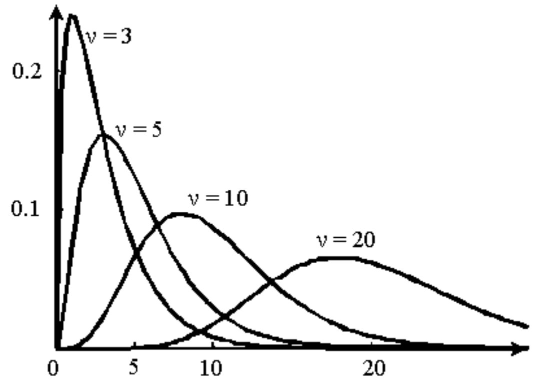
Notation: $\vec{X} = (X_1, X_2, \ldots, X_n)$ where $X_1, \ldots, X_n$ form a random sample (X_i's are from a population distribution).

Sample Mean = $\overline{X}$ = $\frac{X_1 + X_2 + \cdots + X_n}{n}$.

Sample Variance = S^2 = $\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$



Facts

Let X ~ N(μ, σ²) and consider a random sample

1. $E(\overline{X}) = \mu$
2. $E(S^2) = \sigma^2$
3. $\overline{X}$ and S^2 are independent
4. $\overline{X} \sim N(\mu, \sigma^2/n)$
5. $(n-1)S^2$ has a central chi-squared distribution with $n - 1$ degrees of freedom

$X^2(d)$ = chi-squared distribution with d degrees of freedom.
This is a special case of the gamma distribution $\Gamma(\lambda, t)$ where $\lambda = 1/2$, $t = d/2$, $d \in \mathbb{Z}^+$.
(Density function of the gamma distribution is
$f(x) = \frac{1}{\Gamma(t)}\lambda^t x^{t-1} e^{-\lambda x}$ where $x \geq 0$ and $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$.)
In practice, if $Z_1, Z_2, \ldots, Z_d$ are iid's with N(0, 1) distributions, we can obtain $X^2(d)$ distribution by $X = Z_1^2 + Z_2^2 + \cdots + Z_d^2$.

Let $U = X^2(d)$ with pdf $f_U(u) = \frac{1}{\Gamma(\frac{d}{2})}\left(\frac{1}{2}\right)^{\frac{d}{2}} u^{\frac{d}{2}-1} e^{-\frac{u}{2}}$ for $u \geq 0$.

Consider another random variable, V which is independent of U and has $N(0, 1)$ distribution. This independence can occur if

$U = (n-1)\frac{S^2}{\sigma^2}$ and $V = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \sqrt{n} \bullet \frac{\overline{X} - \mu}{\sigma}$. Note that $f_V(v) = \frac{1}{\sqrt{2\pi}} \bullet e^{\frac{-v^2}{2}}$

Let $T = \frac{V}{\sqrt{U/d}}$. The distribution of T is called (student's) t-distribution with d degrees of freedom. For an exercise, show $f_T(t) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi d}\,\Gamma(\frac{d}{2})}\left(1 + \frac{t^2}{d}\right)^{-\frac{d+1}{2}}$.

For $T = \frac{V}{\sqrt{U/d}}$ and $W = U$, we can define $G(u, v) = (v\sqrt{\frac{u}{d}}, u) = (t, w)$.

Hence $u = s_1(t, w) = w$ and $v = s_2(t, w) = t\sqrt{\frac{d}{w}}$.

Since U and V are independent, then

$f_{U,V}(u, v) = f_U(u)f_V(v) = \frac{1}{\Gamma(\frac{d}{2})}\left(\frac{1}{2}\right)^{\frac{d}{2}} u^{\frac{d}{2}-1} e^{-\frac{u}{2}} \bullet \frac{1}{\sqrt{2\pi}} \bullet e^{\frac{-v^2}{2}}$.

And $J(t, w) = \begin{vmatrix} \frac{\partial s_1(t,w)}{\partial t} & \frac{\partial s_2(t,w)}{\partial t} \\ \frac{\partial s_1(t,w)}{\partial w} & \frac{\partial s_2(t,w)}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & \sqrt{\frac{w}{d}} \\ 1 & \frac{t}{\sqrt{d}} \end{vmatrix} = -\sqrt{\frac{w}{d}}$. So then

$f_{T,W}(t, w) = f_{U,V}(w, t\sqrt{\frac{w}{d}}) = \frac{1}{\Gamma(\frac{d}{2})}\left(\frac{1}{2}\right)^{\frac{d}{2}} w^{\frac{d}{2}-1} e^{-\frac{w}{2}} \bullet \frac{1}{\sqrt{2\pi}} \bullet e^{\frac{-t^2 w}{2d}} \bullet \left|-\sqrt{\frac{w}{d}}\right|$.

And $f_T(t) = \int_0^\infty \frac{1}{\Gamma(\frac{d}{2})}\left(\frac{1}{2}\right)^{\frac{d}{2}} w^{\frac{d}{2}-1} e^{-\frac{w}{2}} \bullet \frac{1}{\sqrt{2\pi}} \bullet e^{\frac{-t^2 w}{2d}} \bullet \sqrt{\frac{w}{d}}\, dw$ =

$\frac{\left(\frac{1}{2}\right)^{\frac{d+1}{2}}}{\sqrt{\pi d}\,\Gamma(\frac{d}{2})} \int_0^\infty w^{\frac{d+1}{2}-1} e^{-w\bullet\frac{1}{2}\left(1+\frac{t^2}{d}\right)} dw$.

Let $y = \frac{1}{2}\left(1+\frac{t^2}{d}\right)w$. Then $w = \frac{2d}{d+t^2}y$ and $dw = \frac{2d}{d+t^2}dy$.

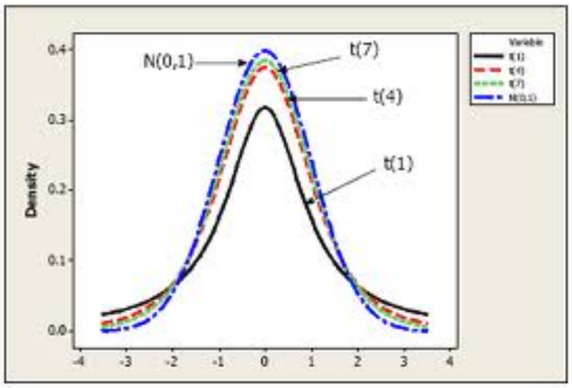
So then $\int_0^\infty w^{\frac{d+1}{2}-1} e^{-w\bullet\frac{1}{2}\left(1+\frac{t^2}{d}\right)} dw$ becomes $\int_0^\infty \left(\frac{2d}{d+t^2}y\right)^{\frac{d+1}{2}-1} e^{-y} \frac{2d}{d+t^2} dy$ =

$\left(\frac{2d}{d+t^2}\right)^{\frac{d+1}{2}} \int_0^\infty y^{\frac{d+1}{2}-1} e^{-y} dy = \left(\frac{2d}{d+t^2}\right)^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)$.

$\therefore \frac{\left(\frac{1}{2}\right)^{\frac{d+1}{2}}}{\sqrt{\pi d}\,\Gamma(\frac{d}{2})} \int_0^\infty w^{\frac{d+1}{2}-1} e^{-\frac{w}{2}\bullet\frac{1+t^2}{d}} dw = \frac{\left(\frac{1}{2}\right)^{\frac{d+1}{2}}}{\sqrt{\pi d}\,\Gamma(\frac{d}{2})} \left(\frac{2d}{d+t^2}\right)^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)$ =

$\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi d}\,\Gamma(\frac{d}{2})} \left(\frac{1}{2}\bullet\frac{1+t^2}{d}\right)^{-\frac{d+1}{2}} \left(\frac{1}{2}\right)^{\frac{d+1}{2}} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi d}\,\Gamma(\frac{d}{2})} \left(1+\frac{t^2}{d}\right)^{-\frac{d+1}{2}}$.

As the degree of freedom increases, the curve becomes more normal.
For $t = 0$, the mean does not exist.
For $t = 1$, the mean exists, but standard deviation does not.
For $t = 2$, the mean and standard deviation exist, but the 3rd moment does not.
And so on.
There is more probability in the tails of a *t*-distribution than in a normal distribution.



The *t*-distribution arises from the sample data as follows: $t = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$ (studentized version of $\overline{X}$).

Simulations
Consider (X_1, X_2) with a bivariate normal distribution such that
$X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$, and $\rho = \text{corr}(X_1, X_2)$.
Compute $P(X_1 < X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f(x_1, x_2)\,dx_1 dx_2$.
If we have a random number generator, to obtain Z_1, Z_2 from $N(0, 1)$ distribution, we can construct X_1, X_2 as follows:
$X_1 = \sigma_1 Z_1 + \mu_1$, and $X_2 = \sigma_2(\rho Z_1 + \sqrt{1+\rho^2}\, Z_2)$.