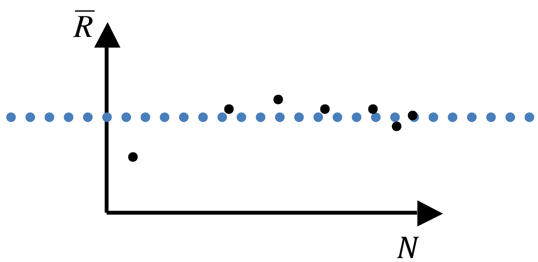
Homework #5:

	$U(0, 1)$			max	min	R (range)
	X_1	X_2	X_3	$X_{(3)}$	$X_{(1)}$	$X_{(3)} - X_{(1)}$
n trials	.012 ⋮	.530 ⋮	.719 ⋮	.719 ⋮	.012 ⋮	.707 ⋮ R_N

As $N \to \infty$, $\overline{R} \to E(R)$ by the law of large numbers.
The same for standard deviation of R and var(R).

Plot the results on a graph



Generating Functions

1. **Probability generating function** (p. 150 (10) Definition)
$G_X(t) = E[t^x]$, X is a random variable, $t \in \mathbb{R}$.
Mostly used for discrete random variables.

2. **Moment generating function (mgf)** (p. 181 (1) Definition)
$M_X(t) = G_X(e^t) = E[(e^t)^X] = E[e^{tX}]$ $t \in \mathbb{R}$ (if it exists)

3. **Cumulant generating functions** (p. 185 Exercise 3):
$S(t) = \ln M_X(t)$

4. **Characteristic function** (p. 182 (2) Definition):
$\varphi(t) = E[e^{itX}]$, $i = \sqrt{-1}$

For $t = 0$, $M_X(0) = 1$, we say that the mgf of a random variable (or its distribution) if $M_X(t) < \infty$ for all $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Examples

1. **Bernoulli (p) distribution** (p. 186 (1) Example):

$M_X(t) = E[e^{tX}] = \sum_{x=0}^{1} e^{tx} P(X = x) = \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x} = \sum_{x=0}^{1} \left(pe^t\right)^x (1-p)^{1-x}$

$= (pe)^{t\bullet 0} \bullet (1-p)^{1-0} + (pe^t)^1 \bullet p^1 = pe^t + (1-p) < \infty$

So $M_X(t)$ exists for all $t \in \mathbb{R}$.

2. **Binomial distribution** $X \sim B(n, p)$ (p. 186 (2) Example):

$E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} \left(pe^t\right)^x (1-p)^{n-x} = (pe^t + (1-p))^n \ \forall\ t \in \mathbb{R}$.

(Recall the identity: $(u + v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}$.)

3. **Poisson distribution**, $X \sim$ Poisson (λ) (p. 151 (17) (d) Example):

pmf: $p(x) = e^{-\lambda} \dfrac{\lambda^x}{x!}$ for $x = 1, 2, \ldots$

$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \dfrac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} e^{tx} \dfrac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \dfrac{\left(\lambda e^t\right)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda\left(e^{e^t}-1\right)}, \forall\ t \in \mathbb{R}$

(Recall the identity $e^a = \sum_{n=0}^{\infty} \dfrac{a^n}{n!}$.)

Theorem

If $X_1, \ldots, X_n$ are iid with mgf $M(t)$ for $t \in D \subseteq \mathbb{R}$, then
$Y = X_1 + \cdots + X_n$ has the mgf $M_Y(t) = [M(t)]^n \ \forall\ t \in \mathbb{R}$.

Proof:

$M_Y(t) = E[e^{t(X_1 + \cdots X_n)}] = E[e^{tX_1} \bullet \cdots \bullet e^{tX_n}] = E[e^{tX_1}] \bullet \cdots \bullet E[e^{tX_n}] = [M(t)]^n$.

Example

$X \sim B(n, p) \Rightarrow X = Z_1 + \cdots + Z_n$, $Z_i \sim$ Bernoulli (p) $\Rightarrow M_X(t) = (pe^t + (1-p))^n \ \forall\ t \in \mathbb{R}$.

Mgf depends on the distribution, not the random variable X.

Convergence of the cdf of a sequence of random variables X_i:

Let $F_n(x)$ be a sequence of cdf's with corresponding mgfs $M_n(t)$.
If $F_n(x) \to F(x)$ pointwise such that $F(x)$ is also a cdf of a random variable X, this is called a weak convergence or convergence in distribution, or convergence in law.

Theorem

If a sequence of mgfs $M_n(t)$ converges to $M(t)\ \forall\ t \in D \subseteq \mathbb{R}$ such that $M(t)$ is mgf of a random variable X, then the corresponding sequence of cdf's F_n also converges to the cdf of X.

Obtaining moments from the mgf:
Continuous case:
Let X be a continuous random variable with pdf $f(x)$. Assume $M_X(t)$ exists on a neighborhood of 0.

Consider $\frac{d}{dt}M_X(t) = \frac{d}{dt}\int e^{tx}f(x)dx = \int \frac{\partial}{\partial t}\left(e^{tx}f(x)\right)dx = \int xe^{tx}f(x)dx = E[Xe^{tX}]$.

Now $\lim_{t\to 0} E[Xe^{tX}] = \lim_{t\to 0}\int xe^{tx}f(x)dx = \int \lim_{t\to 0}\left(xe^{tx}f(x)\right)dx = \int xf(x)dx = E(X)$.

Notation (p. 181) : $M_X'(0) = E(X)$. In general, $M^{(k)}(0) = E[X^k]$.