Mgf: $M_X(t) = E[e^{tX}]$ if it exists $\forall\, t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Recall:
1. Mgf is unique if it exists.
2. Mgf describes a distribution (cdf, pdf, pmf, etc.) not a random variable.
3. It is not guarantee to exist for some distributions (e.g. lognormal dist.)
4. If the mfg's of a sequence of rv's $\{X_n\}_{n\in\mathbb{N}}$ converge to another mgf, say $M_X(t)$ of a random variable X, then the corresponding cdf;s also converge.

If $M_{Xn}(t) \to M_X(t)$ $\forall\, t \in (-\varepsilon, \varepsilon)$ pointwise, then the corresponding cdf's converge, $F_{Xn}(x) \to F_X(x)$ (weak convergence).

Example 1. For $X \sim B(n, p)$, $M_X(t) = (pe^t + (1 - p))^n$ $\forall\, t \in \mathbb{R}$.
Example 2. For $X \sim$ Poisson(λ), $M_X(t) = e^{\lambda(e^t - 1)}$

Application
Let $\lambda = np$. Then n increases as p decreases. Show that the mgf of $B(n, p)$ converges to the mgf of Poisson(λ).
Proof:
$p_n = \lambda/n$. Let $M_{Xn}(t) = \left(\frac{\lambda}{n}e^t + (1 - \frac{\lambda}{n})\right)^n = \left(\frac{\lambda}{n}\left(e^t - 1\right) + 1\right)^n = e^{\lambda(e^t - 1)}$
(since $\left(1 + \frac{a_n}{n}\right)^n \to e^a$ if $a_n \to a$).
Exercise:
Show for a fixed p, $(pe^t + (1 - p))^n$ converges to the mgf of a normal distribution $N(\mu, \sigma^2)$, $\mu = p$, $\sigma^2 = np(1 - p)$.

Example 3. For $X \sim N(\mu, \sigma^2)$, $M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Proof: $\int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2 - 2tx\sigma^2\right)} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2(\mu + \sigma^2 t)x + \mu^2\right)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x - 2(\mu + \sigma^2 t)\right)^2 - \frac{1}{2\sigma^2}\left(\mu^2 - (\mu + \sigma^2 t)^2\right)} dx$$

$$= e^{-\frac{1}{2\sigma^2}\left(\mu^2 - (\mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2)\right)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x - 2(\mu + \sigma^2 t)\right)^2} dx$$

$$= e^{-\frac{1}{2\sigma^2}\left(-2\mu\sigma^2 t - \sigma^4 t^2\right)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x - 2(\mu + \sigma^2 t)\right)^2} dx$$

$$= e^{\left(\mu t - \frac{1}{2}\sigma^2 t^2\right)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x - 2(\mu + \sigma^2 t)\right)^2} dx$$

$$= e^{\left(\mu t - \frac{1}{2}\sigma^2 t^2\right)} \bullet 1.$$

Example 4. For $X \sim \log N(\mu, \sigma^2)$, $M_X(t)$ does not exist.
Proof:
Let $Y = \ln X \sim \log N(\mu, \sigma^2)$. Then $P(X \geq 0) = 1$. And for $x > 0$
$F_X(x) = P(X \leq x) = P(e^Y \leq x) = P(Y \leq \ln x) = F_Y(\ln x)$. So then

$$f_X(x) = \frac{d}{dx} F_Y(\ln x) = f_Y(\ln x) \bullet 1/x = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}.$$

This gives us that $M_X(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} dx$.

Let $g(x, t) = \frac{e^{tx}}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}$. Then we can find a function $h(n, t, x)$ such that $\int_{-\infty}^{\infty} h(t,x)dx = \infty$ and $h(t, x) \leq g(t, x)$.

The moments of lognormal distributions all exist, but there is no mgf of the lognormal.

For $\log N(0, 1)$, $E[X^r] = e^{\frac{r^2}{2}}$.

Non-unique moments

X_1: $\log N(0, 1)$, $f_1(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{(\ln x)^2}{2}}$ for $x > 0$.
X_2: $f_2(x) = f_1(x)[1 + \sin(2\pi \ln x)]$ $0 < x < \infty$.

X_1 and X_2 have the same moments, $E[X_i^r] = e^{\frac{r^2}{2}}$ for $i = 1, 2, \ldots$.
Exercise: Show this is true for any $r = 1, 2, \ldots$.
Try $E(X)$ and $E(X^2)$.

Use mgf to generate the moments:
Example
For $X \sim B(n, p)$, $M_X(t) = (pe^t + (1 - p))^n$, and

$$E[X] = \left.\frac{d}{dt} M_X(t)\right|_{t=0} = \left.\frac{d}{dt}\left(pe^t + (1-p)\right)^n\right|_{t=0} = \left.npe^t\left(pe^t + (1-p)\right)^{n-1}\right|_{t=0} =$$

$$npe^0\left(pe^0 + (1-p)\right)^{n-1} = np.$$

And $$E[X^2] = \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0} = \left.\frac{d}{dt} npe^t\left(pe^t + (1-p)\right)^{n-1}\right|_{t=0} =$$

$$\left.n(n-1)\left(pe^t\right)^2\left(pe^t + (1-p)\right)^{n-2} + \left(npe^t\right)\left(pe^t + (1-p)\right)^{n-1}\right|_{t=0} = n(n-1)p^2 + np.$$

To check for accuracy, we know $\text{var}(X) = np(1 - p)$ and
$E(X^2) - E(X)^2 = n(n - 1)p^2 + np - (np)^2 = np(1 - p)$.

Exercise:

For $X \sim N(\mu, \sigma^2)$, $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. Find $E(X)$, $E(X^2)$, $E(X^3)$.
I may ask this on the final exam.