Midterm Exam 2.1 - Math 230A
Fall 2010

If not otherwise stated every set is a subset of a metric space (X, d).

1. Show that finite unions and any intersections of closed sets are closed. Give an example of a countable union of closed sets which is not closed.

2. Show that:

(a) $|d(x,z) - d(z,y)| \leq d(x,y)$, for all $x, y, z \in X$.

(b) If $A, B \subset X$ are closed then $A \setminus B$ is open relative to A.

(c) If $C, D \subset X$, then

$$int(C) \bigcup int(D) \subseteq int(C \cup D) ,$$

where $int(C)$ denotes the interior of the set C. Show an example where we don't have equality.

3.

(a) Show that if $K \subset X$ is compact, then it is closed and bounded.

(b) Show that if $A, B \subset X$ are compact, then $A \cup B$ and $A \cap B$ are compact.

4.

(a) Give the definition of a perfect set.

(b) Show that a perfect set is uncountable in $\mathbb{R}^n$.

(c) Consider $A = \{x \in \mathbb{Q} \, : \, x > 0 \, , \, 2 < x^2 < 5\}$. Is A perfect in $X = \mathbb{Q}$? How this example relates to part [(b)].

5.

(a) Give the definition of separated sets A and B.

(b) Show that if there exists an $\varepsilon > 0$ such that $d(a,b) \geq \varepsilon$ for all $a \in A$ and $b \in B$ then A and B are separated.

(c) Show that the Cantor set C is totally disconnected, which means that for all $x, y \in C$, $x < y$, there exists $z \notin C$ such that $x < z < y$.

6. Construct a bijective function $f : [0, 2] \to (0, 2)$. Give an explanation of your construction.