Fall 2010

If not otherwise stated every set is a subset of a metric space (X, d).

Part 1 - Definitions

1. Write the definition a countable set.
2. What are the axioms of a metric?
3. Give the definitions of open, closed and compact sets.
4. Give the definition of a Cauchy sequence.
5. Define the one-to-one, onto and bijective functions.
6. Define the absolute convergence and conditional convergence of series.

Part 2 - Theorems - Prove three (3) of the following four (4) theorems:

1. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.
2. If $\{K_n\}_{n\in\mathbb{N}}$ is a collection of nonempty, compact subsets of X with the finite intersection property, then $\cap_{n=1}^{\infty} K_n \neq \emptyset$.
3. Prove that in $X = \mathbb{R}^n$ every Cauchy sequence is convergent.
4. State and prove the Root Test for series.

Part 3 - Exercises - Solve three (3) out of the of the following five (5) exercises:

1.

(a) Prove that if $f : X \to Y$, then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, for all $A, B \subset Y$.
(b) Show that if $card(A) \leq card(B)$ and $card(B) \leq card(C)$, then $card(A) \leq card(C)$ for any three sets A, B, C.

2.

Let $\{x_n\}_{n\geq 1}$ be a sequence in X such that

$$d(x_{n+1}, x_n) \leq \frac{1}{3}\, d(x_n, x_{n-1})\,, \quad \forall\ n \geq 2\,.$$

Show that $\{x_n\}$ is a Cauchy sequence.
What can we say if suppose instead that

$$d(x_{n+2}, x_n) \leq \frac{1}{3}\, d(x_n, x_{n-2})\,, \quad \forall\ n \geq 3?$$

3.

(a) Determine whether the following series are absolutely convergent, conditionally convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{n 3^{n+2}}{5^n}\,, \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}\,, \quad \sum_{n=1}^{\infty} \sin\frac{1}{n}\,.$$

(b) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n\,.$$

4.

Let $a_n \in \mathbb{R}$, $\forall\, n \in \mathbb{N}$.

(a) Prove that if $0 \leq a_{n+1} \leq a_n$, $\forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} n\, a_n = 0$.

(b) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges.

5.

Show that if $\{x_n\} \subset \mathbb{R}$ and $x_n \to x$, then

$$s_n = \frac{x_1 + x_2 + \ldots + x_n}{n} \to x\,.$$