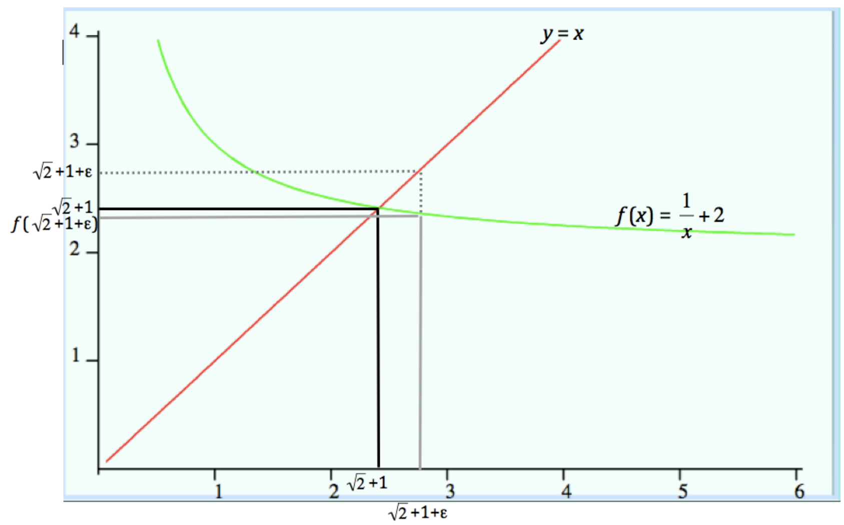
Rudin, Example 1.1, p.2

The equation $p^2 = 2$ is not satisfied by any rational p.
Let $A = \{p \in \mathbb{Q}^+ \mid p^2 < 2\}$. Let $B = \{p \in \mathbb{Q}^+ \mid p^2 > 2\}$.
Let $q = p - \dfrac{p^2 - 2}{p + 2} = \dfrac{2p + 2}{p + 2}$.
If $p \in A$, then $q > p$ and $q \in A$.
If $p \in B$, then $q < p$ and $q \in B$.
$\therefore$ A contains no largest member and B contains no smallest member.

To see what motivates the formula above, consider the function $f(x) = \dfrac{1}{x} + 2$.

Clearly, $f(\sqrt{2}+1) = \sqrt{2}+1$. One can see from the graph below that for sufficiently small ε, $f(\sqrt{2}+1+\varepsilon)$ is closer to $\sqrt{2}+1$ than $\sqrt{2}+1+\varepsilon$ is. Similarly $f(\sqrt{2}+1-\varepsilon)$ is closer to $\sqrt{2}+1$ than $\sqrt{2}+1-\varepsilon$ is. This can be verified using Taylor's Formula.



Note that if $p \approx \sqrt{2}$, then $f(p+1) - 1$ will give us an approximation of $\sqrt{2}$.

And so $\sqrt{2} \approx f(p+1) - 1 = \dfrac{1}{p+1} + 2 - 1 = \dfrac{p+2}{p+1}$.

One can also see from the graph that if $p > \sqrt{2}$, then $f(p+1) < \sqrt{2}+1$, (and if $p < \sqrt{2}$, then $f(p+1) > \sqrt{2}+1$), but we would like to have the reverse inequality. We can remedy this by noting that for $p = \sqrt{2}$, $\sqrt{2}\,(f(p+1)-1) = 2$, hence $\sqrt{2} = 2/(f(p+1)-1) = \dfrac{2p+2}{p+2}$.

This gives us that $p < \sqrt{2} \Rightarrow (f(p+1)-1) > \sqrt{2} \Rightarrow 2/(f(p+1)-1) < \sqrt{2}$.
Similarly, $p > \sqrt{2} \Rightarrow (f(p+1)-1) < \sqrt{2} \Rightarrow 2/(f(p+1)-1) > \sqrt{2}$.

And note that by the identity $p - \dfrac{p^2-2}{p+2} = \dfrac{2p+2}{p+2}$, we have

$p > \sqrt{2} \Rightarrow p^2 - 2 > 0 \Rightarrow \dfrac{p^2-2}{p+2} > 0$ (recall $p \in \mathbb{Q}^+$) $\Rightarrow p > p - \dfrac{p^2-2}{p+2} = \dfrac{2p+2}{p+2}$.

Similarly, $p < \sqrt{2} \Rightarrow p < p - \dfrac{p^2-2}{p+2} = \dfrac{2p+2}{p+2}$.

Thus $p > \sqrt{2} \Rightarrow p > 2/(f(p+1)-1) > \sqrt{2}$ and $p < \sqrt{2} \Rightarrow p < 2/(f(p+1)-1) < \sqrt{2}$, as desired.