Math 230A, Week #1 - Practice Problems

1. Let $A \subset \mathbb{R}$ be nonempty and bounded above. Show that if $\alpha = \sup A \notin A$, then $\forall\ \varepsilon > 0$ the interval $(\alpha - \varepsilon, \alpha)$ contains infinitely many elements of A.
What about the case $\alpha \in A$?

Proof:
Assume $\alpha = \sup A \notin A$. Let $\varepsilon > 0$.

Suppose $(\alpha - \varepsilon, \alpha)$ contains finitely many elements of A.

Then $(\alpha - \varepsilon, \alpha) = \{a_1, a_2, ..., a_n\}$ for some integer $n \geq 0$ where $a_i < a_j$ for $i < j$.

If $n = 0$, then $\alpha - \varepsilon$ is an upper bound of A, contrary to $\alpha = \sup A$.

So then we can assume $n \geq 1$.

Note that $\alpha \notin A \Rightarrow a_n < \alpha$. Let $x \in A$. Then $x \leq a_n$ as $(a_n, \alpha) \cap A = \emptyset$.

So a_n is an upper bound, contrary to $\alpha = \sup A$.

Thus $(\alpha - \varepsilon, \alpha)$ contains infinitely many elements of A.

If $\alpha \in A$, then α may be an isolated point, in which case $(\alpha - \varepsilon, \alpha)$ may be empty.

2. Let $A, B \subset \mathbb{R}$ be bounded sets. Show that
a) $\inf(A + B) = \inf A + \inf B$

Proof:

$A + B = \{a + b | a \in A, b \in B\}$

Let $x \in A + B$. Since $x \in A + B$, then $x = a + b$ for some $a \in A$ and $b \in B$.

And $\inf A \leq a$, and $\inf B \leq b$. Thus $\inf A + \inf B \leq a + b = x$.

Hence $\inf A + \inf B$ is a lower bound of $A + B$, or equivalently,

$\inf A + \inf B \leq \inf(A + B)$.

Suppose $\inf A + \inf B < \inf(A + B)$.
Let $\alpha = \inf A$, let $\beta = \inf B$, let $\gamma = \inf(A + B)$.

Let $\varepsilon = [\gamma - (\alpha + \beta)]/2$.

Note that $\varepsilon > 0$.

Then $\alpha + \varepsilon$ is not a lower bound of A. So $\exists\ x \in A$ such that $\alpha \leq x < \alpha + \varepsilon$.

And $\beta + \varepsilon$ is not a lower bound of B. So $\exists\ y \in B$ such that $\beta \leq y < \beta + \varepsilon$.

$x + y \in A + B$ and $\alpha + \beta \leq x + y < \alpha + \beta + 2\varepsilon = \alpha + \beta + [\gamma - (\alpha + \beta)] = \gamma$.

Thus, $x + y$ is a lower bound, contrary to $\gamma = \inf(A + B)$.

2. b) $\sup(A + B) = \sup A + \sup B$

Proof:

Let $x \in A + B$. Since $x \in A + B$, then $x = a + b$ for some $a \in A$ and $b \in B$.

And $\sup A \geq a$, and $\sup B \geq b$. Thus $\sup A + \sup B \geq a + b = x$.

Hence $\sup A + \sup B$ is an upper bound of $A + B$, or equivalently,

$\sup A + \sup B \geq \sup(A + B)$.

Suppose $\sup A + \sup B > \sup(A + B)$.

Let $\alpha = \sup A$, let $\beta = \sup B$, let $\gamma = \sup(A + B)$.

Let $\varepsilon = [(\alpha + \beta) - \gamma]/2$.

Note that $\varepsilon > 0$.

Then $\alpha - \varepsilon$ is not an upper bound of A. So $\exists\, x \in A$ such that $\alpha \geq x > \alpha - \varepsilon$.

And $\beta - \varepsilon$ is not an upper bound of B. So $\exists\, y \in B$ such that $\beta \geq y > \beta - \varepsilon$.

$x + y \in A + B$ and $\alpha + \beta \geq x + y > \alpha + \beta - 2\varepsilon = \alpha + \beta + [(\alpha + \beta) - \gamma] = \gamma$.

Thus, $x + y$ is an upper bound of $A + B$, contrary to $\gamma = \sup(A + B)$.

c) Is $\sup(A \bullet B) = \sup A \bullet \sup B$?

No.

Proof:

Let $A = (-2, 2)$, and let $B = (-5, 1)$. Then $A \bullet B = (-10, 10)$. So then $\sup A = 2$, $\sup B = 1$, and $\sup A \bullet B = 10 \neq 2 = \sup A \bullet \sup B$.

d) $\sup(-A) = -\inf A$

Proof:

Let $A \subset \mathbb{R}$ such that $A \neq \varnothing$ and A is bounded.

Since $\mathbb{R}$ has the least upper bound property and A is bounded below, then

$\alpha = \inf(A)$ exists. Let $-A = \{-x \mid x \in A\}$.

We will show (i) $\sup(-A)$ exists; and

(ii) $\exists\, \rho \in R$ such that $\forall\, \varepsilon > 0$ $-\alpha \geq \rho > \alpha - \varepsilon$; hence $-\alpha = \sup(-A)$.

(i) Since $\forall\, x \in A$, $\alpha \leq x$, then $\forall\, -x \in -A$, $-\alpha \geq -x$.

So $-A$ is bounded above by $-\alpha$. Thus $\sup(-A)$ exists.

(ii) $\forall\, \varepsilon > 0$ $\exists\, \gamma \in A$ such that $\alpha \leq \gamma < \alpha + \varepsilon$ by definition of infimum of A. Let $\rho = -\gamma$.

Thus $-\alpha \geq \rho > -\alpha - \varepsilon$.

$\therefore$ $-\alpha = \sup(-A)$, hence $\alpha = \inf(A) = -\sup(-A)$, as desired.

2. e) $\sup(A - B) = \sup A - \inf B$

Proof:
We can write $\sup(A - B) = \sup(A + (-B))$.

By part (b) $\sup(A + (-B)) = \sup A + \sup(-B)$.

By part (d) $\sup(-B) = -\inf B$.

Thus $\sup(A - B) = \sup A - \inf B$.

f) $\sup(2A) = 2\sup A$

Proof:
Let $x \in 2A$, then $x = 2a$ for some $a \in A$. And $\sup(A) \geq a \Rightarrow 2\sup A \geq 2a = x$.

So $2\sup A$ is an upper bound for $\sup(2A)$, hence $2\sup A \geq \sup(2A)$.

Suppose $2\sup A > \sup(2A)$.

Let $\alpha = \sup A$, and let $\beta = \sup(2A)$.

Let $\varepsilon = 2\alpha - \beta$.

Then $\exists\, w \in A$ such that $2\alpha \geq w > 2\alpha - \varepsilon$ and $\exists\, y \in 2A$ such that $\beta \leq y < \beta - \varepsilon$.

Thus $\alpha \geq 2w > \alpha - \varepsilon = \beta$. So then $2w$ is an upper bound for $2A$,

contrary to $\beta = \sup(2A)$.

g) inf A ≤ sup A

Proof:
Let $\alpha = \inf A$, and let $\beta = \sup A$.

Then $\forall\, x \in A$, $\alpha \leq x \leq \beta$, hence $\alpha \leq \beta$.

g) $\inf A$ and $\sup A$ are unique. In other words every set A has at most one glb or lub.

Proof:
Suppose $\alpha = \inf A$ and $\alpha' = \inf A$.

Then $\forall\, x \in A$, $\alpha \leq x$. Since α' is a lower bound of A, then $\alpha' \leq \alpha$.

And since α is a lower bound of A, then $\alpha \leq \alpha'$.

Thus $\alpha = \alpha'$ by definition "≤". $\therefore$ $\inf A$ is unique.

Similarly $\sup A$ is unique.

2. h) $\inf(A \cup B) = \min\{\inf A, \inf B\}$

Proof:
Let $\alpha = \inf A$, let $\beta = \inf B$, and let $\gamma = \min\{\alpha, \beta\}$.

Then $\forall\ x \in A \cup B$, $\alpha \le x$ or $\beta \le x$, hence $\gamma \le x$.

Thus γ is a lower bound of $A \cup B$.

Then $\exists\ y \in A$ such that $\alpha \le y < \alpha + \varepsilon$

or $\exists\ y \in B$ such that $\beta \le y < \beta + \varepsilon$

$\therefore\ \gamma \le y < \gamma + \varepsilon$. So then y is not a lower bound of $A \cup B$.

Hence $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

i) $\sup(A \cup B) = \max\{\sup A, \sup B\}$

Proof:
Let $\alpha = \sup A$, let $\beta = \sup B$, and let $\gamma = \max\{\alpha, \beta\}$.

Then $\forall\ x \in A \cup B$, $\alpha \ge x$ or $\beta \ge x$, hence $\gamma \ge x$.

Thus γ is an upper bound of $A \cup B$.

Then $\exists\ y \in A$ such that $\alpha \ge y > \alpha + \varepsilon$

or $\exists\ y \in B$ such that $\beta \ge y > \beta + \varepsilon$

$\therefore\ \gamma \ge y > \gamma + \varepsilon$. So then y is not an upper bound of $A \cup B$.

Hence $\sup(A \cup B) = \max\{\inf A, \inf B\}$.

What about $A \cap B$?

Since $A \cap B$ could be empty, then no comparison can be made between inf/sup of A, B and $A \cap B$.