1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof:

Let $r \in \mathbb{Q}$, $r \neq 0$, and let $x \in \mathbb{R} - \mathbb{Q}$. Then $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.

Assume that $r + x$ and rx are rational.

Then $r + x = \frac{c}{d}$ and $rx = \frac{j}{k}$ for some $c, d, j, k \in \mathbb{Q}$ such that $d, k \neq 0$.

Since b, d and hence $bd \neq 0$, we have $x = \frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{bd} \in \mathbb{Q}$ as $cb - ad, bd \in \mathbb{Z}$.

And since $k, r \neq 0$, hence $a \neq 0$, we have $x = \frac{j}{k} \bullet \frac{1}{r} = \frac{j}{k} \bullet \frac{1}{\frac{a}{b}} = \frac{jb}{ka} \in \mathbb{Q}$ as $jb, ad \in \mathbb{Z}$.

Thus x is rational, contrary to the hypothesis that x is irrational.

$\therefore$ $r + x$ and rx are irrational.

2. Prove that there is no rational number whose square is 12.

Proof:

Suppose there is. Then $\exists$ r $\in \mathbb{Q}$ such that $r^2 = 12$ where $r = \frac{a}{b}$ such that gcd(a, b) = 1,

for some $a, b \in \mathbb{Z}$, $b \neq 0$.

And $\left(\frac{a}{b}\right)^2 = 12$ which gives us that $a^2 = 12b^2 = 2^2 \bullet 3b^2$. Thus $3|a^2$.

By the Fundamental Theorem of Arithmetic, $a = p_1^{n_1} \bullet p_2^{n_2} \cdots p_m^{n_m}$, positive integer powers of primes, so $a^2 = p_1^{2n_1} \bullet p_2^{2n_2} \cdots p_m^{2n_m}$.

Since $3 \mid a^2$, we have $p_i = 3$ for some i, hence $3|a$ and $3^2 \mid a^2$.

This gives us that $3^2 \mid 2^2 \bullet 3b^2$, and so $2^2 \bullet 3b^2 = 3^2d$ for some integer d.

Thus $2^2b^2 = 3d$. And so by Euclid's Lemma we have $3 \mid b^2$.

So then, by parallel argument to $3|a$, we have $3|b$, hence $\gcd(a, b) \geq 3$, contrary to our assumption that $\gcd(a, b) = 1$.

$\therefore$ There is no rational number whose square is 12.

3. Prove Proposition 1.15.
(a) If $x \neq 0$, and $xy = xz$, then $y = z$.

Proof:
Let $x, y, z \in F$ where $x \neq 0$.

First note that by M4, $1y = y$ and $1z = z$.

Since $x \neq 0$ then by M5 $1/x$ exists and $(1/x) \bullet x = 1$.

Thus we have $y = 1 \bullet y = (1/x) \bullet x \bullet y = (1/x) \bullet x \bullet z = 1 \bullet z = z$.

(b) If $x \neq 0$, and $xy = x$, then $y = 1$.

Proof:
Let $x, y \in F$ where $x \neq 0$.

First note that by M4, $1y = y$ and $1x = x$.

Since $x \neq 0$ then by M5 $1/x$ exists and $(1/x) \bullet x = 1$.

Thus we have $y = 1 \bullet y = (1/x) \bullet x \bullet y = (1/x) \bullet x = 1 \bullet x = x$.

(c) If $x \neq 0$, and $xy = 1$, then $y = 1/x$.

Proof:
Let $x, y \in F$ where $x \neq 0$.
First note that by M4, $1y = y$ and $(1/x) \bullet 1 = 1/x$.
Since $x \neq 0$ then by M5 $1/x$ exists and $(1/x) \bullet x = 1$.
Thus we have $y = 1 \bullet y = (1/x) \bullet x \bullet y = (1/x) \bullet 1 = 1/x$.

(d) If $x \neq 0$, then $1/(1/x) = x$.

Proof:
Let $x \in F$ where $x \neq 0$.

First note that by M4, $1 \bullet x = x$.

Since $x \neq 0$ then by M5 $1/x$ exists and $(1/x) \bullet x = 1$.

We know $1/x \neq 0$, otherwise $(1/x) \bullet x = 0$.

So then M4 also gives us that $1/(1/x) = [1/(1/x)] \bullet 1$.

Thus we have $1/(1/x) = [1/(1/x)] \bullet 1 = [1/(1/x)] \bullet (1/x) \bullet x = 1 \bullet x = x$.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof:
Let $E \subset S$ such that $E \neq \emptyset$, bounded below and bounded above.

Let α be a lower bound of E, and let β be an upper bound of E.

Then $\forall\, x \in E$, $\alpha \leq x \leq \beta$, hence $\alpha \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf(A) = -\sup(-A)$.

Proof:
Let $A \subset \mathbb{R}$ such that $A \neq \emptyset$ and A is bounded below.

Since $\mathbb{R}$ has the least upper bound property and A is bounded below, then

$\alpha = \inf(A)$ exists. Let $-A = \{-x |\ x \in A\}$.

We will show (i) $-A$ is bounded above by $-\alpha$; and

(ii) $\exists\, \rho \in \mathbb{R}$ such that $\forall\, \varepsilon > 0$ $-\alpha \geq \rho > \alpha - \varepsilon$; hence $-\alpha = \sup(-A)$.

(i) Since $\forall\, x \in A$, $\alpha \leq x$, then $\forall\, -x \in -A$, $-\alpha \geq -x$.

So $-A$ is bounded above by $-\alpha$. Thus $\sup(-A)$ exists.

(ii) $\forall\, \varepsilon > 0$ $\exists\, \gamma \in A$ such that $\alpha \leq \gamma < \alpha + \varepsilon$ by definition of infimum of A. Let $\rho = -\gamma$.

Thus $-\alpha \geq \rho > -\alpha - \varepsilon$.

$\therefore$ $-\alpha = \sup(-A)$, hence $\alpha = \inf(A) = -\sup(-A)$, as desired.

6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that $\left(b^m\right)^{\frac{1}{n}} = \left(b^p\right)^{\frac{1}{q}}$.

Hence it makes sense to define $b^r = \left(b^m\right)^{\frac{1}{n}}$.

Proof:
Let $r \in \mathbb{Q}$ such that $r = m/n = p/q$, for some integers m, n, p, q where $n, q > 0$.

First note that $mq = mqn(1/n) = (m/n)qn = (p/q)qn = p\bullet 1\bullet n = pn$.

Then $[(b^m)^{1/n}]^{pn} = b^{pm} = [(b^p)^{1/q}]^{mq}$.

Thus, by the uniqueness assertion of Theorem 1.21, $\left(b^m\right)^{\frac{1}{n}} = \left(b^p\right)^{\frac{1}{q}}$.

6. (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Proof:
We first define $x^0 = 1$ and $x^{-1} = 1/x$ for any nonzero x. Assume $b^{r+s} = b^r b^s$ if $r, s \in \mathbb{Z}$.

Let $r, s \in \mathbb{Q}$. Then $r = a/b$ and $s = c/d$ for some $a, b, c, d \in \mathbb{Z}$ where $b, d \neq 0$.

Then $b^r = b^{a/b}$, $b^s = b^{c/d}$ and

$$\left(b^r b^s\right)^{bd} = \left(b^{\frac{a}{b}} b^{\frac{c}{d}}\right)^{bd} = \left(b^{\frac{a}{b}}\right)^{bd}\left(b^{\frac{c}{d}}\right)^{bd} = b^{ad} b^{bc} = b^{ad+bc} = \left(b^{\frac{ad+bc}{bd}}\right)^{bd} = \left(b^{r+s}\right)^{bd}.$$

Thus, by the uniqueness assertion of Theorem 1.21, $b^r b^s = b^{r+s}$.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$.
Prove that $b^r = \sup B(r)$ when r is rational.
Hence it makes sense to define $b^x = \sup B(x)$ for every real x.

Proof:
Claim (1): $\forall\, n \in \mathbb{Z}$ such that $b > 1$, we have $b > 1 \Rightarrow b^{1/n} > 1$.

By the identity $b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \cdots + b^{n-n})$ $(*)$ we have also

$b - 1 = (b^{1/n} - 1)(b^{(n-1)/n} + b^{(n-2)/n} + \cdots + b^{(n-n)/n})$.

So then, if $0 < b - 1$, we have $0 < b^{1/n} - 1$ as clearly $0 < b^{(n-1)/n} + b^{(n-2)/n} + \cdots + b^{(n-n)/n}$.

Thus $b^{1/n} > 1$.

Claim(2): $\forall\, m, r \in \mathbb{Q}$ such that $m \leq r$, $b^m \leq b^r$.

$m \leq r \Rightarrow r - m \geq 0$. And $r - m \in \mathbb{Q}$. So then $r - m = a/c$ for some $a, c \in \mathbb{Z}^+$.

Note that by the identity $(*)$ we have $b - 1 > 0 \Rightarrow b^n - 1 > 0$, hence $b > 1 \Rightarrow b^a > 1$.

Thus, by Claim (1), $b > 1 \Rightarrow b^a > 1 \Rightarrow b^{r-m} = b^{a/c} = (b^a)^{1/c} \geq 1$. And so $b^r \geq b^m$.

Let $r \in \mathbb{Q}$ and let $B(r) = \{b^t | t \in \mathbb{Q} \text{ and } t \leq r\}$. By Claim (2), $B(r)$ is bounded above by b^r.

And since $b^r \in B(r)$, then $b^r = \sup B(r)$.

Let $x \in \mathbb{R}$. To show we can define $b^x = \sup B(x)$, we note that by the Archimedean

Property of $\mathbb{Q}$ $\exists\, m \in \mathbb{Q}$ such that $m > x$. Thus $B(x)$ is bounded above by b^m.

Thus $\sup B(x)$ exists.

And by the density property,$\forall\, r \in \mathbb{Q}$ such that $r < x$, $\exists\, q \in \mathbb{Q}$ such that $r < q \leq x$.

By Claim(2), $\forall\, r < x$, $b^r < b^q$, hence b^r is not an upper bound of $B(x)$.

So if we define $b^x = \sup B(x)$, the conditions for $\sup B(x)$ are satisfied.

6. (d) Prove that $b^{x+y} = b^x b^y$.

Proof:
By part (c), we have
$b^x = \sup B(x)$, $b^y = \sup B(y)$, and b^{x+y}.
To show $b^{x+y} = b^x b^y$ we first prove the following lemma:

Let $A \subseteq \mathbb{R}^+$, let $B \subseteq \mathbb{R}^+$, then $\sup AB = \sup A \bullet \sup B$.

Proof:
Let $\alpha = \sup A$, let $\beta = \sup B$, and let $\gamma = \sup AB$.
Let $x \in AB$, then $x = ab$ for some $a \in A$, $b \in B$.
Since $a \leq \alpha$, $b \leq \beta$, and $a, b > 0$, then $x = ab \leq \alpha\beta$. Thus $\alpha\beta$ is an upper bound of AB.
So then $\alpha\beta \geq \gamma$.
Suppose $\alpha\beta > \gamma$.
Then $\exists\, x \in \mathbb{R}^+$ such that $\gamma \leq x < \alpha\beta$.
Let $\varepsilon = \alpha\beta - x$. Then $x = \alpha\beta - \varepsilon$.
Let $\delta = \varepsilon/(\alpha + \beta)$.
Since $\alpha = \sup A$, then $\exists\, a \in A$ such that $\alpha - \delta < a \leq \alpha$, and
since $\beta = \sup B$, then $\exists\, b \in B$ such that $\beta - \delta < b \leq \beta$.
So then $= (\alpha - \delta)(\beta - \delta) < ab \leq \alpha\beta$ which implies

$$\begin{aligned} \gamma \leq x &= \alpha\beta - \varepsilon \\ &< \alpha\beta - \varepsilon + [\varepsilon/(\alpha + \beta)]^2 \\ &= \alpha\beta - \delta(\alpha + \beta) + \delta^2 \\ &= (\alpha - \delta)(\beta - \delta) < ab \leq \alpha\beta. \end{aligned}$$

However, $ab \in AB$ and $ab \leq \gamma\ \forall\, a \in A$ and $\forall\, b \in B$, clearly a contradiction
$\therefore$ $\alpha\beta = \gamma$ as desired.

Since $B(x)$ and $B(y)$ are subsets of $\mathbb{R}^+$, then we can apply this lemma to obtain
$b^{x+y} = b^x b^y$.

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$.
(This is called the logarithm of y to the base b.)
Complete the following outline.
(a) For any positive integer n, $b^n - 1 \geq n(b - 1)$.

Proof:
Let $n \in \mathbb{Z}^+$. Since $b > 1 \Rightarrow b^{n-i} \geq 1\ \forall\ i \leq n$, we have
$b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \cdots + b^{n-n}) \geq (b - 1)(1 + 1 + \cdots + 1) = n(b - 1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

Proof:
Let $a = b^n$. By Claim (1) of Exercise 6, part (c), $a^{1/n} > 1$.
Thus by part (a) $a - 1 \geq n(a^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

Proof:
We first note that $t > 1 \Rightarrow t - 1 > 0$ and $b > 1 \Rightarrow b^{\frac{1}{n}} > 1 \Rightarrow b^{\frac{1}{n}} - 1 > 0$.

So then
$$n > \frac{b-1}{t-1} \geq \frac{n(b^{\frac{1}{n}} - 1)}{t-1}$$
$$\Rightarrow\ 1 > \frac{1}{n} \bullet \frac{n(b^{\frac{1}{n}} - 1)}{t-1} = \frac{b^{\frac{1}{n}} - 1}{t-1}$$
$$\Rightarrow\ 1 \bullet (t - 1) > b^{\frac{1}{n}} - 1$$
$$\Rightarrow\ t - 1 > b^{\frac{1}{n}} - 1$$
$$\Rightarrow\ t > b^{1/n}.$$

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n.

Proof:
First note that $b^w < y \Rightarrow 1 < yb^{-w}$.
Let $n > \dfrac{b-1}{yb^{-w} - 1}$, then by part (c), $b^{1/n} < yb^{-w}$, thus $b^{w+(1/n)} < y$.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

Proof:
Note that $b^w > y \Rightarrow b^w y^{-1} > 1$.
Let $n > \dfrac{b-1}{b^w y^{-1} - 1}$, then by part (c), $b^{1/n} < b^w y^{-1}$, thus $b^{w-(1/n)} > y$.

7. (f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup(A)$ satisfies $b^x = y$.

Proof:
Let $y \in \mathbb{R}$, let $A = \{w \mid b^w < y\}$, and let $x = \sup(A)$.
Suppose $b^x < y$. Then by the Archimedean Property, $\exists\, n > \dfrac{b-1}{yb^{-x}-1}$, hence $b^{x-1/n} > y$.
So $x - 1/n$ is an upper bound of A. But this is a contradiction to our assumption that x is the least upper bound of A.
Suppose $b^x > y$. Then by the Archimedean Property, $\exists\, n > \dfrac{b-1}{b^x y^{-1}-1}$, hence $b^{x+1/n} < y$.
So then $x + 1/n \in A$, but $x + 1/n > x$, the least upper bound of A, a contradiction.
$\therefore\ b^x = y$.

(g) Prove that this x is unique.

Proof:
Suppose $\exists\, d \in \mathbb{R}$ such that $d = \sup(A)$.
If $x < d$, then $b^{d-x} < 1 \Rightarrow b^x < b^d$.
If $x > d$, then $b^{x-d} < 1 \Rightarrow b^x > b^d$.
$\therefore\ x = d$.