HW02, Practice Problems, Assigned Thursday, 9/9/10

1. Read and understand Proposition 1.18 and its proof.
Proposition 1.18 The following statements are true in every ordered field.
(a) If $x > 0$, then $-x < 0$, and vice versa.
(b) If $x > 0$ and $y < z$, then $xy < xz$.
(c) If $x < 0$ and $y < z$, then $xy > xz$.
(d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
(e) If $0 < x < y$ then $0 < 1/y < 1/x$.

Proof:
(a) If $x > 0$ then $0 = -x + x > -x + 0 = -x$.
If $x < 0$ then $0 = -x + x < -x + 0 = -x$.

(b) $x > 0$ and $y < z \Rightarrow z - y > y - y = 0$ and $x(z - y) > 0$, by Definition 1.17 part (ii)
Thus $xz = x(z - y) + xy > 0 + xy = xy$.

(c) By part (a), $x < 0 \Rightarrow -x > 0$. And $y < z \Rightarrow z - y > 0$, hence $(-x)(z - y) > 0$.
By Prop 1.16(c) $-[x(z - y)] = (-x)(z - y)$.
Thus $-[x(z - y)] > 0$; and by part (a) $x(z - y) < 0$. Hence $xz < xy$.

(d) If $x > 0$, then by Definition 1.17 part (ii), $x^2 > 0$.
If $x < 0$, then $-x > 0$ by part (a) and $(-x)^2 > 0$. Since $x^2 = (-x)^2$ by Proposition 1.16(d), then $x^2 > 0$.
To show $1 > 0$, we note that $1 \neq 0$ by definition of field, and $1 = 1^2$.
So then $x \neq 0 \Rightarrow x^2 > 0$ gives us that $1 = 1^2 > 0$.

(e) Claim: $0 < y \Rightarrow 0 < 1/y$.
If $1/y < 0$, then $1 = y \bullet 1/y < 0$ by part (c), hence $1/y > 0$. If $1/y = 0$, then $1 = y \bullet 1/y = 0$ by Proposition 1.16 (a). Thus $1/y > 0$.
Similarly $0 < x \Rightarrow 0 < 1/x$.
Thus $(1/x)(1/y) > 0$ by Definition 1.17 part (ii).
So then $0 < x < y \Rightarrow x \bullet (1/x)(1/y) = 1/y < 1/x = (1/x)(1/y) \bullet y$.

2. Read Definition 1.24 of the field of Complex Numbers $\mathbb{C}$.
(a) Ex. 8, Ch. 1 says that no order can be defined which turn $\mathbb{C}$ into an ordered field. Explain.
If $\mathbb{C} = \{x + yi\colon x, y \in \mathbb{R}, i^2 = -1\}$ can be made into an ordered field, then it must follow the field axioms as well as the structure of the order.
In particular, $1 > 0$. And exactly one of $i = 0$, $i < 0$, or $i < 0$ may hold.
If $i = 0$, then $i^4 = i^3 \bullet i = -1 \bullet 0 \neq 1 = (-1)(-1) = i^2 \bullet i^2$, a contradiction.
If $i < 0$, then $1 = i^4 < 0 \bullet i^3 = 0$, a contradiction.
If $i > 0$, then $-1 = i \bullet i > i \bullet 0 = 0$, hence $0 = -1 + 1 > 0 + 1 = 1$, a contradiction.
$\therefore$ No order can be defined which turn $\mathbb{C}$ into an ordered field.

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2. (b) Consider the following partial order on $\mathbb{C}$: $(a, b) \le (c, d)$ if $a \le c$ and $b \le d$. What is the set of positive elements?
The set of positive elements = $\{(a, b)\colon a > 0 \text{ and } b > 0\}$.

Does this order satisfy the two axioms of Definition 1.17?
No.

Axiom (ii) fails.

Let $(a, b), (c, d) > (0, 0)$. Then $(a, b)(c, d) = (ac - bd, ad + bc)$.

If $ac < bd$, then $ac - bd < 0$, hence $(a, b)(c, d) < (0, 0)$.

This contradicts axiom (ii) in which $x, y > 0 \Rightarrow xy > 0$.

Hint: For exercises 3, 4 you may use the prime factorization theorem.
3. Let n be a positive integer which is not a perfect square. Prove that $\sqrt{n}$ is irrational.

Proof:
Let $n \in \mathbb{N}$ such that n is not a perfect square. Then $n = p_1^{n_1} \bullet p_2^{n_2} \bullet\bullet\bullet\, p_{m_0}^{n_{m_0}}$ where the p_i's are distinct primes and, without loss of generality, n_1 is odd.
Assume $\sqrt{n} \in \mathbb{Q}$. Then $\sqrt{n} = a/b$, or equivalently, $b^2 n = a^2$.
We can write $a = a_1^{s_1} \bullet a_2^{s_2} \bullet\bullet\bullet\, a_{m_2}^{s_{m_2}}$ and $b = b_1^{r_1} \bullet b_2^{r_2} \bullet\bullet\bullet\, b_{m_1}^{r_{m_1}}$ where the a_i's and b_i's are distinct primes. So then

$$\left(b_1^{r_1} \bullet b_2^{r_2} \bullet\bullet\bullet\, b_{m_1}^{r_{m_1}}\right)^2 \bullet\ p_1^{n_1} \bullet p_2^{n_2} \bullet\bullet\bullet\, p_{m_0}^{n_{m_0}} = \left(a_1^{s_1} \bullet a_2^{s_2} \bullet\bullet\bullet\, a_{m_2}^{s_{m_2}}\right)^2.$$

$$= \left(b_1^{r_1} \bullet b_2^{r_2} \bullet\bullet\bullet\, b_{m_1}^{r_{m_1}} \bullet p_1^{t_1} \bullet p_2^{t_2} \bullet\bullet\bullet\, p_{m_0}^{t_{m_0}}\right)^2.$$

$$b_1^{2r_1} \bullet b_2^{2r_2} \bullet\bullet\bullet\, b_{m_1}^{2r_{m_1}} \bullet p_1^{n_1} \bullet p_2^{n_2} \bullet\bullet\bullet\, p_{m_0}^{n_{m_0}} = b_1^{2r_1} \bullet b_2^{2r_2} \bullet\bullet\bullet\, b_{m_1}^{2r_{m_1}} \bullet p_1^{2t_1} \bullet p_2^{2t_2} \bullet\bullet\bullet\, p_{m_0}^{2t_{m_0}}.$$

$$p_1^{n_1} \bullet p_2^{n_2} \bullet\bullet\bullet\, p_{m_0}^{n_{m_0}} = p_1^{2t_1} \bullet p_2^{2t_2} \bullet\bullet\bullet\, p_{m_0}^{2t_{m_0}}.$$

By the Fundamental Theorem of Arithmetic, for each i, $n_i = 2t_i$, hence $n_1 = 2t_1$.
However, n_1 is odd, clearly a contradiction.
$\therefore\ \sqrt{n}$ is irrational.

4. Prove that there is no rational number r such that $2^r = 3$.

Proof:
Assume $\exists\, r \in \mathbb{Q}$ such that $2^r = 3$. Then $r = a/b$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.
So then $2^{a/b} = 3$. Thus $2^a = (2^{a/b})^b = 3^b$. Since $2 \neq 3$ and both are primes, then we have two distinct prime factorizations representing the same number.
This is in contradiction to the Fundamental Theorem of Arithmetic.

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5. Solve Exercise 12, Chapter 1.
If $z_1, \ldots, z_n$ are complex, prove that $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$.

Proof:
We will prove the result by induction on n.

Let $n = 1$, then $|z_1| \le |z_1|$.

Let $n > 1$ and assume $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$.

Let $x = z_1 + z_2 + \cdots + z_n$.

Then $z_1 + z_2 + \cdots + z_n + z_{n+1} = x + z_{n+1}$.

Since $\forall\, w, z \in C$, $|z| = (z\bar{z})^{1/2}$ by definition, $\overline{z + w} = \bar{z} + \bar{w}$ by thm 1.31(a),

$|zw| = |z||w|$ by thm 1.33(c), and $|\bar{z}| = |z|$ by thm 1.33(b), then

$|x + z_{n+1}|^2 = (x + z_{n+1})(\bar{x} + \bar{z}_{n+1}) = x\bar{x} + x\bar{z}_{n+1} + \bar{x}z_{n+1} + \bar{z}_{n+1}z_{n+1}$

$= |x|^2 + 2\text{Re}(x\bar{z}_{n+1}) + |z_{n+1}|^2$

$\le |x|^2 + 2|x\bar{z}_{n+1}| + |z_{n+1}|^2$

$= |x|^2 + 2|x||\bar{z}_{n+1}| + |z_{n+1}|^2$

$= |x|^2 + 2|x||z_{n+1}| + |z_{n+1}|^2$

$= (|x| + |z_{n+1}|)^2$.

$\therefore$ $|x + z_{n+1}| \le |x| + |z_{n+1}|$.

$\therefore$ $|z_1 + z_2 + \cdots + z_n + z_{n+1}| \le |z_1| + |z_2| + \cdots + |z_{n+1}|$.

Hence $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$for all n.

6. Solve Exercise 13, Chapter 1.
If x, y are complex, prove that $||x| - |y|| \le |x - y|$.

Proof:
Let $x, y \in \mathbb{C}$.

$|x| = |x - y + y| \le |x - y| + |y|$ by the triangle inequality.

Thus $|x| - |y| \le |x - y|$.

If $|x| \ge |y|$, then $|x| - |y| = ||x| - |y||$.

If $|x| < |y|$, then $|x| - |y| = -||x| - |y|| \le ||x| - |y||$.

In either case, $||x| - |y|| \le |x - y|$.

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7. Solve Exercise 17, Chapter 1 in the case of complex numbers.
Prove that $|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$ if $x \in \mathbb{C}$ and $y \in \mathbb{C}$.
Interpret this geometrically, as a statement about parallelograms.

Proof:
Let $x \in \mathbb{C}$ and $y \in \mathbb{C}$.

so the distributive property holds. Thus

$$\begin{aligned} |x + y|^2 + |x - y|^2 &= (x + y)(\bar{x} + \bar{y}) + (x - y)(\bar{x} - \bar{y}) \\ &= x \bullet \bar{x} + x \bullet \bar{y} + y \bullet \bar{x} + y \bullet \bar{y} + x \bullet \bar{x} - x \bullet \bar{y} - y \bullet \bar{x} + y \bullet \bar{y}, \\ &= 2|x|^2 + 2|y|^2. \end{aligned}$$

Since $x + y$ represents the diagonal of the parallelogram formed by two vectors and $x - y$ represents the other diagonal, then this identity verifies that the sum of the squares of the diagonals of a parallelogram equals the sum of twice the squares of the sides.

8. Prove Bernoulli's inequaltiy:
If $x > -1$ and $x \neq 0$, then $(1 + x)^n > 1 + nx$ for each $n \geq 2$.

Proof:
Let $x \in \mathbb{R}$ such that $x > -1$ and $x \neq 0$. We will prove the result by induction on n.

Let $n = 2$. Then $(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$.

Let $n > 2$ and assume $(1 + x)^n > 1 + nx$. Then

$$\begin{aligned} (1 + x)^{n+1} &= (1 + x)(1 + x)^n \\ &= (1 + x)^n + x(1 + x)^n \\ &> (1 + x)(1 + nx) \text{ since } x > -1 \Rightarrow 1 + x > 0 \\ &= 1 + (n + 1)x + nx^2 \\ &> 1 + (n + 1)x \text{ since } n > 0 \text{ and } x^2 > 0. \end{aligned}$$

$\therefore$ $(1 + x)^n > 1 + nx$ for each $n \geq 2$.